June 1974

LRP 84/74

NONLINEAR WAVES AND FLUCTUATIONS IN PLASMAS

M.V. Goldman

Centre de Recherches en Physique des Plasmas

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

June 1974 LRP 84/74

NONLINEAR WAVES AND FLUCTUATIONS IN PLASMAS

M.V. Goldman

Department of Astro-Geophysics*, University of Colorado
Boulder, Colorado

LECTURE I

A. INTRODUCTION

Starting from a microscopic description, we review the linear theory of field fluctuations about ensemble-average values near collective resonances. Following the concepts of nonlinear optics, we will then compare the coherent and incoherent nonlinear current expansions in the fields. Matrix elements will be evaluated for a wide variety of interactions, using the fluid description with an isothermal equation of state. For the coherent fields the parametric approximation will be developed, and some linear parametric instabilities of coherent and incoherent waves discussed. The analogous weak turbulence theory will be developed for incoherent waves, first in the absence of a coherent field, and then in its presence. A simple theory of saturation for the parametric decay instability will be presented. Finally, some of the idea of trapping in coherent fields and associated "sideband" instabilities will be discussed.

---

*On visit at the Centre de Recherches en Physique des Plasmas.

B. KLIMONTOVICH FORMALISM

1. Microscopic equations:

$\underline{r}_i^s(t) \equiv$ trajectory of i^{th} particle of species s, accelerating due to total microscopic fields $\underline{\mathcal{E}}_{MIC}$ and $\underline{B}_{MIC}$ (including self-consistent and external fields). The trajectories obey Newton's equations,

$$\ddot{\underline{r}}_i^s(t) = \frac{q_s}{m_s}\left[\underline{\mathcal{E}}_{MIC}(\underline{r}_i^s(t),t) + \frac{\dot{\underline{r}}_i^s(t)}{c} \times \underline{B}_{MIC}(\underline{r}_i^s,t)\right], \quad i = 1, \ldots N$$

plus initial conditions (N particles)

These equations are equivalent to the exact microscopic Liouville equation for the microscopic one-particle distribution function:

$$\mathcal{F}^s_{MIC}(\underline{v},\underline{r},t) = \sum_{i=1}^{N} \delta^3(\underline{r}-\underline{r}_i^s(t))\,\delta^3(\underline{v}-\underline{v}_i^s(t)):$$

$$\left\{\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{r}} + \frac{q_s}{m_s}\left[\underline{\mathcal{E}}_{MIC} + \frac{\underline{v}}{c}\times\underline{B}_{MIC}\right]\cdot\frac{\partial}{\partial \underline{v}}\right\} F^s_{MIC}(\underline{v},\underline{r},t) = 0$$

Comments:

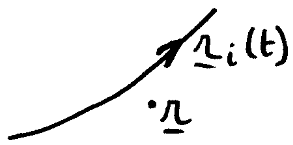
a) This equation is exact, and not to be confused with collisionless Boltzmann or Vlasov equations in which collisional effects are ignored as an approximation.

b) The equation is nonlinear in the sense of the dependence of $\mathcal{F}^s_{MIC}(\underline{v},\underline{r},t)$ on $\underline{\mathcal{E}}_{MIC}$ and $\underline{B}_{MIC}$.

c) The coordinates $\underline{v}$ and $\underline{r}$ are Eulerian. That is, they are in the fixed laboratory frame of reference. The coordinates $\underline{r}_i(t)$ and $\underline{v}_i(t)$, are Lagrangian. That is, they move along with the particles relative to the laboratory frame.

d) The distribution function, $\mathcal{F}^s_{MIC}(\underline{v},\underline{r},t)$, is a phase space ($\underline{r}$-$\underline{v}$ space) density. Any description of a system of discrete particles in terms of a density requires the use of delta functions. If $\mathcal{F}^s_{MIC}$ is integrated over some volume in $\underline{r}$-$\underline{v}$ space at a time t, the result is the total number of particles in that volume at that time.

e) The microscopic quantities such as $\mathcal{F}^s_{MIC}$, $\underline{\mathcal{E}}_{MIC}$, and $\underline{B}_{MIC}$ are rapidly varying in time. For example, $\mathcal{E}_{MIC}(\underline{r},t)$ may get large when the orbit $\underline{r}^s_i(t)$ of a particular particle i passes close to the position $\underline{r}$ at time t



The microscopic charge and current densities are:

$$\rho^s_{MIC}(\underline{r},t) = q_s \sum_{i=1}^{N} \delta^3(\underline{r}-\underline{r}^s_i(t)) \equiv q_s \int d^3v\, \mathcal{F}^s_{MIC}(\underline{v},\underline{r},t)$$

$$\underline{j}^s_{MIC}(\underline{r},t) = q_s \sum_{i=1}^{N} \dot{\underline{r}}^s_i\, \delta^3(\underline{r}-\underline{r}^s_i(t)) = q_s \int d^3\underline{v}\; \underline{v}\; \mathcal{F}^s_{MIC}(\underline{v},\underline{r},t)$$

Just as the fields give rise to these current and charge densities through Newton's or the Liouville equations, the (self-consistent part of the) fields are produced by those sources through Maxwell's equations:

$$\underline{\nabla}\cdot\underline{\mathcal{E}}_{MIC} = 4\pi\left(\rho^e_{MIC}+\rho^i_{MIC}+\rho_{Ext}\right) \qquad \underline{\nabla}\cdot\underline{B}_{MIC} = 0$$

$$\underline{\nabla}\times\underline{\mathcal{E}}_{MIC} = -\frac{1}{c}\frac{\partial}{\partial t}\underline{B}_{MIC} \qquad \underline{\nabla}\times\underline{B}_{MIC} = \frac{4\pi}{c}\left(\underline{j}^e_{MIC}+\underline{j}^i_{MIC}+\underline{j}_{EXT}\right)+\frac{1}{c}\frac{\partial\underline{\mathcal{E}}_{MIC}}{\partial t}$$

These equations are far too detailed to be useful, especially since their solution requires a complete knowledge of the initial positions and velocities of all the particles. For that reason, some kind of average is usually performed. This may be a volume, time, phase or ensemble-average. We will treat only ensemble-averages.

2. Ensemble-averages

If a is a stochastic variable such as ρ^e_{MIC}, j^i_{MIC}, $\mathcal{E}_{MIC}$, or B_{MIC}, it depends indirectly on the phase space configuration $\underline{r}_1(o)$, $\underline{v}_1(o)$, $\underline{r}_2(o)$, etc., at some initial time t = 0, since Newton's Law is deterministic, given a complete set of initial conditions. Let $\langle a \rangle = A(\underline{r},t)$, be an ensemble-average over initial coordinates. Note, therefore, that the $\langle\ \rangle$ operation commutes with all the Eulerian coordinates, such as $\underline{r}$, $\underline{v}$, t. Specifically,

$$\langle a \rangle \equiv \int d r_1^e(o) \ldots d r_N^e(o)\, d v_1^e \ldots d v_N^e\, d r_1^i \ldots d r_N^i\, d v_1^i \ldots d v_N^i \; D(r_1^e(o) - v_N^i(o))\, A,$$

where the N-particle initial distribution function is usually taken to be some form of non-interacting "equilibrium", such as

$$D \propto e^{-\sum_i T_i / \theta}$$

, where T_i is the kinetic energy of the i^{th} particle. Interacting fields are generally "turned on slowly" and/or treated perturbatively. We then define the average or "coherent" quantities,

$$\rho^s = \langle \rho^s_{MIC} \rangle, \quad \underline{E} = \langle \mathcal{E}_{MIC} \rangle, \quad f^s = \langle F^s_{MIC} \rangle$$
$$j^s = \langle j^s_{MIC} \rangle, \quad \underline{B} = \langle B_{MIC} \rangle,$$

and the fluctuating, or "incoherent" quantities,

$$\rho^f_s = \rho^s_{MIC} - \rho^s, \quad \underline{\mathcal{E}} = \underline{\mathcal{E}}_{MIC} - \underline{E} \quad \mathcal{F}_s = F^s_{MIC} - f^s$$
$$j^f_s = j^s_{MIC} - j^s \quad \underline{b} = \underline{B}_{MIC} - \underline{B}$$

The ensemble averages of each of the "incoherent" quantities vanishes.

3. Coherent and incoherent equations

It is easy to show that the coherent and incoherent quantities obey the following equations which are still exact:

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{r}}\right) f^s = - \frac{\partial}{\partial \underline{v}} \cdot \left[\langle \underline{a}^s \mathcal{F}^s \rangle + \underline{A}^s f^s \right] \qquad (1)$$

$$\left(\frac{\partial}{\partial t}+\underline{v}\cdot\frac{\partial}{\partial \underline{r}}\right)\mathfrak{F}^s=-\frac{\partial}{\partial \underline{v}}\cdot\left[\underline{a}^s f^s+\underline{A}^s\mathfrak{F}^s+\underline{a}^s\mathfrak{F}^s-\langle\underline{a}^s\mathfrak{F}^s\rangle\right] \tag{2}$$

where the incoherent acceleration $\underline{a}^s$ is $\underline{a}^s=\underline{a}^s_{MIC}-\underline{A}^s$, and

$$\underline{a}^s_{MIC}=\frac{q_s}{m_s}\left\{\underline{\mathcal{E}}_{MIC}(\underline{r},t)+\frac{\underline{v}}{c}\times\underline{B}_{MIC}(\underline{r},t)\right\},\ \text{and}\ \underline{A}^s\equiv\langle\underline{a}^s_{MIC}\rangle \tag{3}$$

The fields obey Maxwell's equations, which may be combined into a wave equation for $\underline{\mathcal{E}}$ or $\underline{E}$, and written as

$$-c^2\underline{\nabla}(\underline{\nabla}\cdot\underline{\mathcal{E}})+c^2\nabla^2\underline{\mathcal{E}}-\frac{\partial^2}{\partial t^2}\underline{\mathcal{E}}=+4\pi\frac{\partial}{\partial t}\left(\underline{j}^f_e+\underline{j}^f_i\right) \tag{4}$$

$$\left.\begin{array}{ll}\underline{\nabla}\cdot\underline{\mathcal{E}}=4\pi\left(\rho^f_e+\rho^f_i\right) & \text{, or, equivalently,}\\ \underline{\nabla}\cdot\underline{j}^f+\frac{\partial}{\partial t}\rho^f=0 & \text{(continuity eq}^{\underline{n}}\text{)}\end{array}\right\} \tag{5}$$

The magnetic field may always be found from the electric field, by

$$\underline{\nabla}\times\underline{\mathcal{E}}=-\frac{1}{c}\frac{\partial\underline{B}}{\partial t} \tag{6}$$

Finally, for the coherent fields,

$$-c^2\underline{\nabla}(\underline{\nabla}\cdot\underline{E})+c^2\nabla^2\underline{E}-\frac{\partial^2}{\partial t^2}\underline{E}=4\pi\frac{\partial}{\partial t}\left(\underline{j}^e+\underline{j}^i+\underline{j}_{EXT}\right) \tag{7}$$

$$\underline{\nabla}\cdot\underline{E}=4\pi\left(\rho^e+\rho^i\right),\ \text{or}\ \ \underline{\nabla}\cdot\underline{j}+\frac{\partial}{\partial t}\rho=0 \tag{8}$$

$$\underline{\nabla}\times\underline{E}=-\frac{1}{c}\frac{\partial\underline{B}}{\partial t} \tag{9}$$

4. Closure in the linear approximation (Coherent waves)

Linear coherent waves: In equation (1) in the collisionless approximation $\langle \underline{a}^s \mathcal{J}^s \rangle$ is ignored. Alternatively, velocity moments of equation (1) are taken, and a binary collisional model of $\langle \underline{a}^s \mathcal{J}^s \rangle$ adopted. (Chap. Enskog Proc.). Both procedures rely on the smallness of the mean potential energy compared to the mean kinetic energy*. The former is used for frequencies larger than the collision frequency; the latter for frequencies much less than the collision frequency . Either way the resulting set of equations is still nonlinear in A^s. An expansion of the current $\underline{j}$ in powers of $\underline{E}$ is generally carried out, and only the linear term retained. Initial value or boundary value information in the current may be lumped with $\underline{j}_{EXT}$. We therefore assume or find a "linear constitutive relation",

$$\underline{j}(\underline{r},t) = \int d^3r' dt' \, \underline{\underline{\sigma}}(\underline{r},\underline{r}';t,t') \cdot \underline{E}(\underline{r}',t') \tag{10}$$

This must be nonlocal in space and time, generally, since moving charges allow a field acting at one space-time point to act as a source for a polarization current at another. The conductivity $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_e + \underline{\underline{\sigma}}_i$ receives a contribution from each species, in general. With (10) inserted in (7) and assuming $\underline{j}_{EXT}$ and boundary conditions are known, solution of the resulting integro-differential equation gives the linear properties of self-consistent fields, including wave propagation.

5. Normal modes in infinite, homogeneous, stationary plasmas

In this case, $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{r}-\underline{r}';t-t')$, and equation (7) may be solved by Fourier transforming in space, and carrying out a one-sided Fourier transform in time:

$$f(\omega) = \int_0^\infty dt \, e^{i\omega t} f(t)$$

*proportional to the plasma parameter, k_D^3 / n, in equilibrium.

$$f(t) = \int_{-\infty+i\Gamma}^{+\infty+i\Gamma} \frac{d\omega}{2\pi}\, e^{-i\omega t} f(\omega) \quad \Gamma > \operatorname{Im}\omega_{POLE},$$

where ω_{POLE} is the singularity of $f(\omega)$ with the largest imaginary part.

$$g(\underline{k}) = \int d^3\underline{r}\; e^{-i\underline{k}\cdot\underline{r}}\, g(\underline{r})$$

$$g(\underline{r}) = \int \frac{d^3\underline{k}}{(2\pi)^3}\, e^{i\underline{k}\cdot\underline{r}}\, g(\underline{k})$$

Then, the Fourier transform of the constitutive relation is

$$\underline{j}(\underline{k},\omega) = \underline{\underline{\sigma}}(\underline{k},\omega)\cdot\underline{E}(\underline{k},\omega) \tag{11}$$

and the wave equation, (7) becomes,

$$\underline{\underline{M}}(\underline{k},\omega)\cdot\underline{E}(\underline{k},\omega) = -4\pi i\, \underline{j}^{EXT}(\underline{k},\omega), \tag{12}$$

where

$$M_{\ell n} = \frac{c^2 k_\ell k_n}{\omega} - \frac{c^2 k^2}{\omega}\delta_{\ell n} + \omega\left(\delta_{\ell n} + \frac{4\pi i\,\sigma_{\ell n}}{\omega}\right) \tag{13}$$

The combination in paranthesis is generally called the dielectric tensor,

$$\epsilon_{\ell n}(\underline{k},\omega) \equiv \delta_{\ell n} + \frac{4\pi i}{\omega}\sigma_{\ell n}(\underline{k},\omega), \tag{14}$$

sometimes it is written as $\delta_{\ell n} + 4\pi\chi_{\ell n}$, which gives us the following definition of the susceptibility tensor:

$$\chi_{\ell n}(\underline{k},\omega) = \frac{i}{\omega}\sigma_{\ell n}(\underline{k},\omega) \tag{15}$$

For later reference we note that (15) implies

$$\underline{\underline{\sigma}}(\underline{r}-\underline{r}', t-t') \equiv \frac{\partial}{\partial t} \underline{\underline{\chi}}(\underline{r}-\underline{r}', t-t') \qquad (16)$$

The waves that can propagate in the plasma are most simply formed in the normal mode analysis, when the driving term $\underline{j}^{EXT}$ in equation (12) is set equal to zero. The normal modes are obtained by finding the zeroes of the determinant of M:

$$\det \underline{\underline{M}}(\underline{k}, \omega) = 0$$

Often $\underline{\underline{M}}$ is or can be made, exactly or approximately, diagonal by an appropriate choice of coordinates. For example, in the absence of a D.C. magnetic field, in the cold plasma approximation,

$$\epsilon_{ij} = \delta_{ij}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \equiv \delta_{ij}\ \epsilon(\omega) \qquad (17)$$

Choosing a coordinate system in which $\hat{z} \parallel \underline{k}$,

$$\underline{\underline{M}} = \begin{pmatrix} \omega\epsilon - c^2k^2/\omega & 0 & 0 \\ 0 & \omega\epsilon - c^2k^2/\omega & 0 \\ 0 & 0 & \omega\epsilon \end{pmatrix} \qquad (18)$$

The two transverse polarizations then have dispersion relations $c^2k^2 = \omega^2\epsilon$, and the longitudinal Langmuir wave dispersion relation is given by $\omega\epsilon = 0$. The representation in (18) is correct (provided $\epsilon_{xx} = \epsilon_{yy} \equiv \epsilon_T$ and $\epsilon_{zz} \equiv \epsilon_L$) even with better approximations for ϵ_L and ϵ_T which may, for example, include an ion species, temperature effects, velocity-space interactions, collisional effects, etc. In general, diagonalizing $\underline{\underline{M}}$ does not guarantee one normal mode for each polarization. For example when $T_e \gg T_i$ there are Langmuir and ion-acoustic waves associated with E_z. Often, however, we can designate one kind of wave for a diagonal element m of a diagonalized matrix $\underline{\underline{M}}$ by making the "resonant approximation":

Let $\omega_c = \omega_r - i\gamma$ be a particular complex root of $m(\omega) = 0$. Then, when $|\gamma| \ll |\omega_r|$, we expand about ω_r:

$$0 = m(\omega_c) = m(\omega_r) - i\gamma \frac{\partial}{\partial\omega} m(\omega)\Big|_{\omega=\omega_r} + \text{higher order terms}$$

Equating real and imaginary parts,

$$\mathrm{Re}\, m(\omega_r) + \gamma \frac{\partial\, \mathrm{Im}\, m(\omega_r)}{\partial\omega_r} = 0 \,, \qquad \gamma = \frac{\mathrm{Im}\, m}{\frac{\partial \mathrm{Re}\, m}{\partial\omega}}\Bigg|_{\omega-\omega_r} \tag{19}$$

assuming $\dfrac{\gamma\, \partial \mathrm{Im}}{\partial\omega}\Big|_{\omega=\omega_r} \ll \mathrm{Re}\, m(\omega_r)$ (guaranteed by $\gamma \ll \omega_r$),

this gives us an algorithm for finding ω_r and γ from $m(\omega)$ when the mode is resonant $(|\gamma| \ll |\omega_r|)$. Next, expand about ω_c:

$$m(\omega) = 0 + (\omega - \omega_r + i\gamma)\frac{\partial \mathrm{Re}\, m}{\partial\omega}\Big|_{\omega=\omega_r} \tag{20}$$

(assumes $\mathrm{Im}\, m \ll \mathrm{Re}\, m$). This is called the resonant approximation to $m(\omega)$ about a particular normal mode.

6. Linear theory of incoherent waves

The conductivity $\underline{\underline{\sigma}}^s$ in the collisionless approximation results from equation (1) after setting $\langle \underline{a}^s \mathfrak{f}^s \rangle = 0$, and expanding $f^s = f_0^s + f_1^s$, where f_0^s is the distribution function for the background plasma (independent of the self-consistant field). For the homogeneous, stationary plasma we have considered, it can depend only on velocity. The corresponding approximation in equation (2), when no coherent fields are present* (A = 0) is to set $\underline{a}^s \mathfrak{F}^s = \langle \underline{a}^s \mathfrak{F}^s \rangle$ (collisionless approximation and partial linearization). The structure of (1) and (2) is then similar, and we may expand $\mathfrak{F}^s = \mathfrak{F}^s_o + \mathfrak{F}^s_1$, where $\mathfrak{F}^s_o$ corresponds to spontaneous emission, such as Cerenkov emission, and $\mathfrak{F}^s_1$ is linear in $\mathcal{E}$ and $\mathcal{B}$.

$\mathfrak{F}_o$ is a solution to

$$\left[\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla}\right] \mathfrak{F}_o = 0$$

when there is no coherent background field.

The solution corresponding to Cerenkov emission is

$$\mathfrak{F}_o(\underline{r}, \underline{v}, t) = \sum_i \delta^3(\underline{r} - \underline{r}_i(t))\, \delta^3(\underline{v} - \dot{\underline{r}}_i(t)) - \left\langle \sum_i \delta^3(\underline{r} - \underline{r}_i(t)) \cdot \delta^3(\underline{v} - \dot{\underline{r}}_i(t)) \right\rangle,$$

where $\underline{r}_i(t)$ is the (straight-line) orbit of a particle in the absence of external fields:

$$\underline{r}_i(t) = \underline{r}_i(0) + \underline{v}_i(0)\, t$$

This can be a basis for computing Cerenkov emission, by averaging over a non-interacting initial ensemble.

---

*Sometimes a background field $\underline{A}_o$ is split off and included on the left side of the f - equation or the $\mathfrak{F}$. equation. f_o and $\mathfrak{F}_o$ then include all orders in this field. A D.C. magnetic field is an example !

Then, we obtain the incoherent analogue to equation (12), that is

$$\underline{\underline{M}}(\underline{k},\omega)\cdot\underline{\mathcal{E}}(\underline{k},\omega) = -4\pi i\, \underline{j}_0(\underline{k},\omega), \tag{21}$$

where $\underline{j}_0$ is the total fluctuating spontaneous emission (as opposed to polarization) current,

$$\underline{j}_0(\underline{k},\omega) = \sum_{s=e,i} q_s \int d^3\underline{v}\; \underline{v}\; \mathfrak{F}_0^s(\underline{v},\underline{k},\omega) \tag{22}$$

In the diagonal, resonant approximation, corresponding to a particular normal mode we may write equation (21) as

$$\underline{\mathcal{E}} = -\frac{4\pi i\, \underline{j}_0^{\,\ell}}{(\omega-\omega_\ell+i\gamma)\dfrac{\partial\, \mathrm{Re}\, m}{\partial\omega}} = -\frac{4\pi i\, \underline{j}_0^{\,\ell}(\underline{k},\omega)}{m\,(\underline{k},\omega)} \tag{23}$$

The measured quantities associated with fluctuations are correlation functions. If A(r,t) and B(r't') are two fluctuating physical quantities, although $\langle a\rangle = \langle B\rangle = 0$, the correlation function

$$C\,(r\,r';\,t\,t') \equiv \langle a(r,t)\; B(r't')\rangle$$

is generally non-vanishing. In a homogeneous stationary (steady-state) plasma in the random phase approximation,

$$C = C(\underline{r}-\underline{r}'; t-t')$$

(This is a <u>consequence</u> of linearization, and an approximation otherwise). It is then easy to show that the Fourier transforms are given by

$$\langle a(\underline{k},\omega)\, B(\underline{k}',\omega')\rangle = C(\underline{k},\omega)\,\delta_{\underline{k},-\underline{k}'}\,\delta_{\omega,-\omega'}\, VT$$
$$= C(\underline{k},\omega)(2\pi)^4\,\delta(\underline{k}+\underline{k}')\,\delta(\omega+\omega'), \quad (24)$$

where $$C(\underline{k},\omega) \equiv \frac{\langle a(\underline{k},\omega)\, B(-\underline{k},-\omega)\rangle}{VT}, \quad (25)$$

The expression proportional to the large volume V and time T is for a system finite in time and space (so that $\underline{k}$ and ω are discrete) . The limit as $VT \to \infty$ should always be understood.

For real B, $B(-\underline{k}, -\omega) = B(\underline{k},\omega)^*$. When B = A, we have an autocorrelation function. Using (23), in more general form, we therefore find a linear expression for the steady-state field spectral function,

$$I(\underline{k},\omega) \equiv \frac{\langle |\varepsilon(\underline{k},\omega)|^2\rangle}{VT} \quad (26)$$

$$I(\underline{k},\omega) = \frac{(4\pi)^2\, S_o(\underline{k},\omega)}{|m|^2}, \quad (27)$$

where the spontaneous emission source term is

$$S_o(\underline{k},\omega) = \frac{\langle |j_o^f(\underline{k},\omega)|^2\rangle}{VT} \quad (28)$$

In the resonance approximation (as in equation (23)), we may approximate $|m|^{-2}$ by

$$|m|^{-2} = \frac{\gamma}{\gamma\left|\frac{\partial}{\partial\omega}\,\mathrm{Re}\,m\right|^2_{\omega=\omega_\eta}\left[(\omega-\omega_\eta)^2+\gamma^2\right]} \longrightarrow \frac{\pi}{\gamma\left|\frac{\partial\,\mathrm{Re}\,m}{\partial\omega}\right|^2_{\omega=\omega_\eta}}\,\delta(\omega-\omega_k(k))$$

This enables us to define the spectral function associated with a particular mode of a particular polarization,

$$I(\omega_k(k)) \equiv 2\int_{\text{resonance}} \frac{d\omega}{2\pi}\, I(\underline{k},\omega)$$

$$= \frac{(4\pi)^2}{\gamma\left|\frac{\partial\,\mathrm{Re}\,m}{\partial\omega}\right|^2_{\omega=\omega_\eta}}\, S_0(\underline{k},\omega_k(k)) \qquad (29)$$

The factor 2 in the definition* comes from the fact that there are resonances at $\pm\,\omega_\eta$. For completeness, we state below the expression for S_0 corresponding to Cerenkov emission:

$$S_0(\underline{k},\omega) = 2\pi\,\frac{\omega^2}{k^2}\sum_{s=e,i} q_s^2 \int d^3\underline{v}\;\delta(\omega-\underline{k}\cdot\underline{v})\, f_0^s(\underline{v}) \qquad (30)$$

Suggested homework:

1. Prove the validity of the microscopic Liouville equation using Newton's equations.
2. Derive equations 1-9, by averaging.
3. Derive an expression for $\underline{\underline{\epsilon}}$ from the collisionless Boltzmann equation.
4. Derive equation (30) by ensemble-averaging Cerenkov emission from a set of non-interacting particles (equivalent to test particle method).

---

*This factor is somewhat arbitrary and is not always used in the literature.

LECTURE II

June 6, 1974

A. REVIEW AND SUPPLEMENT TO LECTURE I

1. We have assumed an infinite, homogeneous, stationary, steady state plasma. Then we have Fourier-space Maxwell equations for both the ensemble-average electric field, $\underline{E}(\underline{r},t)$, and the incoherent field, $\mathcal{E}(\underline{r},t)$, whose ensemble-average vanishes:

$$\left(c^2\frac{k_\ell k_m}{\omega}-\frac{c^2k^2}{\omega}\delta_{\ell m}+\omega\delta_{\ell m}\right)\mathcal{E}_m(\underline{k},\omega)=-4\pi i\left[j_\ell^{ef}(\underline{k},\omega)+j_\ell^{if}(\underline{k},\omega)\right]$$

$$\left(c^2\frac{k_\ell k_m}{\omega}-\frac{c^2k^2}{\omega}\delta_{\ell m}+\omega\delta_{\ell m}\right)E_m(\underline{k},\omega)=-4\pi i\left[j_\ell^{e}(\underline{k},\omega)+j_\ell^{i}(\underline{k},\omega)+j_\ell^{EXT}(\underline{k},\omega)\right]$$

The solution of these equations requires a constitutive relation ; that is, an expression for the coherent current $\underline{j}^s$ in terms of the coherent field $\underline{E}$ and/or an expression for the incoherent current $\underline{j}^{sf}$ in terms of the incoherent field, $\underline{\mathcal{E}}$. To calculate either of these currents one can start from the exact thermal ensemble kinetic equations (1) and (2) of the previous lecture for the ensemble-average (coherent) distribution function f^s and for the incoherent distribution function, $\mathfrak{F}^s$, whose ensemble-average vanishes. We assume the distribution function f^s is identical to distribution functions often defined by others in terms of spatial averages.

2. We indicated in equation (24) two forms for the Fourier transform of the correlation function,

$$C(\underline{r}-\underline{r}';t-t')=\langle a(\underline{r},t)\,B(\underline{r}',t')\rangle$$

These are proven as follows: Suppose we take Fourier transforms over a box V and time internal T assuming periodic boundary conditions. Then, in the limit $VT \rightarrow \infty$ we can recover the usual definitions:

$$a(\underline{k},\omega) = \int_V d^3r \int_0^T dt \, e^{-i(\underline{k}\cdot\underline{r} - \omega t)} \, a(\underline{r},t)$$

$$a(\underline{r},t) = \sum_{\underline{k},\omega} \frac{1}{VT} \, e^{i(\underline{k}\cdot\underline{r} - \omega t)} \, a(\underline{k},\omega)$$

$$\rightarrow \int \frac{d^3\underline{k}}{(2\pi)^3} \frac{d\omega}{2\pi} \, e^{i(\underline{k}\cdot\underline{r} - \omega t)} \, a(\underline{k},\omega)$$

With these periodic boundary conditions, $\underline{k}$ and ω are discrete. For example, $\omega = \frac{2\pi n}{T}$, so that behaviour at t is duplicated at t + T . Taking the transform with respect to $\underline{r}$,t and $\underline{r}'$,t' of the correlation function, we obtain

$$\langle a(\underline{k},\omega) \, B(\underline{k}',\omega') \rangle = \int_V d^3r \int_V d^3r' \int_0^T dt \int_0^T dt'$$

$$e^{-i(\underline{k}\cdot\underline{r} - \omega t)} \, e^{-i(\underline{k}'\cdot\underline{r}' - \omega' t')} \, C(\underline{r}-\underline{r}'; t-t')$$

Now change to coordinates $\underline{r} - \underline{r}'$ and $\frac{\underline{r} + \underline{r}'}{2}$ in place of $\underline{r}$, $\underline{r}'$ and to t - t' and $\frac{t + t'}{2}$, in place of t, t'. The Jacobian of the transformation is one. The result is

$$\langle a(\underline{k},\omega) \, B(\underline{k}',\omega') \rangle = C\left(\frac{\underline{k}-\underline{k}'}{2}, \frac{\omega-\omega'}{2}\right) \times$$

$$\int_V \frac{d^3(\underline{r}+\underline{r}')}{2} \int_0^T \frac{d(t+t')}{2} \, e^{-i\left[(\underline{k}+\underline{k}')\cdot\left(\frac{r+r'}{2}\right) - (\omega+\omega')\left(\frac{t-t'}{2}\right)\right]}$$

The integral is equal to $VT\,\delta_{\underline{k},-\underline{k}'}\,\delta_{\omega,-\omega'}$, so

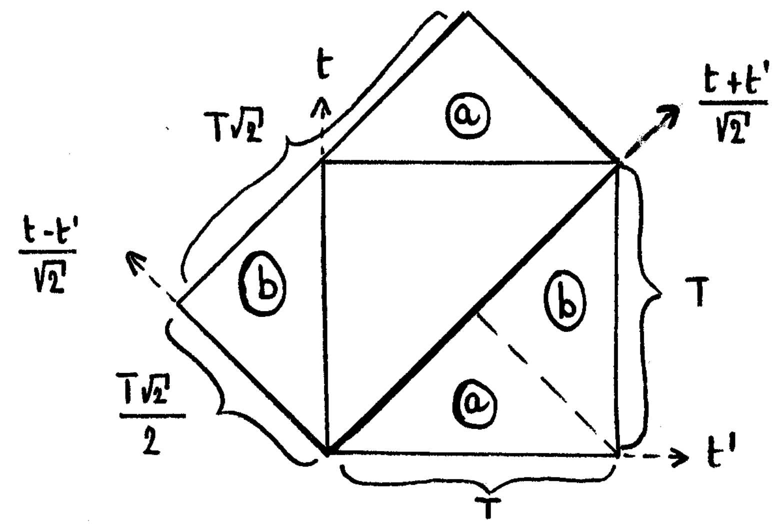
$$\langle \mathcal{A}(\underline{k},\omega)\,\mathcal{B}(\underline{k}',\omega')\rangle = \mathcal{C}(\underline{k},\omega)\,VT\,\delta_{\underline{k},-\underline{k}'}\,\delta_{\omega,-\omega'}$$

In the limit as $VT \to \infty$ the integral becomes

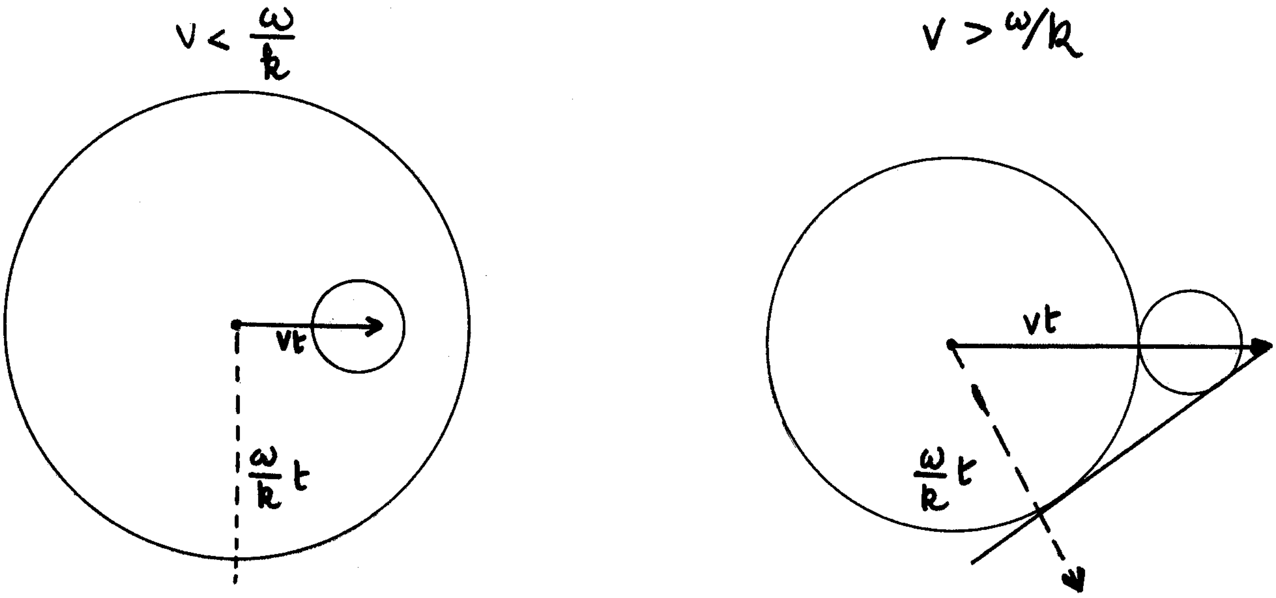
$$(2\pi)^4\ \delta^3(\underline{k}+\underline{k}')\ \delta(\omega+\omega'),$$

which proves the desired result. Note that the effect of the coordinate transformation is a rotation, followed by a compression of one axis and an expansion of the other. For example, consider the t and t' integrations, illustrated below. A 45° rotation followed by an integration over the rectangle bounded by $0 \leq \frac{t-t'}{\sqrt{2}} \leq \frac{T\sqrt{2}}{2}$ and by $0 \leq \frac{t+t'}{\sqrt{2}} \leq T\sqrt{2}$ gives the same result as over the original square, because of the periodic boundary conditions. The new limits are exactly equal to $0 \leq t-t' \leq T$ and $0 \leq \frac{t+t'}{2} \leq T$.

Rotation by 45°:



3. We make some additional comments about the meaning of Cerenkov emission in a plasma. (Equation (30) of the last lecture). Note, the Cerenkov emission of radiation by a single electron moving in a straight line orbit requires that the electron be moving faster than the phase speed, $\frac{\omega}{k}$, of the emitted wave. This can be seen from a Huyghens construction of emitted spherical wave fronts. Constructive interference requires $V > \frac{\omega}{k}$ as shown below



The radiation is emitted at an angle θ to the trajectory given by $\cos\theta = \frac{\frac{\omega}{k}t}{vt} = \frac{\omega}{kv}$. If $\hat{\underline{k}}$ is the direction of propagation of the wave front and $\hat{\underline{v}}$ the velocity direction, then $\underline{k}\cdot\underline{v} = \omega$ is the Cerenkov condition. This can only be satisfied if $V > \omega/k$. The delta function, $\delta(\omega - \underline{k}\cdot\underline{v})$ in equation (30) expresses the Cerenkov condition. However, in that case $\underline{k}$ is the fixed Fourier transform variable, and the velocities are distributed according to the distribution function $f_0^s(v)$. Only those particles for which $V > \omega/k$ can contribute. Thus, one can show that electromagnetic waves cannot be Cerenkov emitted, because $\omega(k)/k = \sqrt{c^2 + \omega_p^2/k^2} > c$. Electrostatic Langmuir waves, on the other hand <u>can</u> be emitted, and it is this spontaneous emission which corresponds to Landau damping when the latter is viewed as a competition between absorption and induced emission, semiclassically.

4. Equation (27), together with (30) can be related to fluctuation-dissipation theorems. In equilibrium, in a collisionless electron plasma, it can be shown directly for longitudinal waves that

$$S_0(\underline{k},\omega) = \frac{k_B T}{2\pi}\,\mathrm{Im}\; m(\underline{k},\omega) = \frac{k_B T}{2\pi}\,\omega\,\mathrm{Im}\,\epsilon(\underline{k},\omega)$$

Thus, equation (27) can be written as

$$I(\underline{k},\omega) = -\frac{(4\pi)^2}{2\pi}\frac{k_B T}{\omega}\,\mathrm{Im}\,\frac{1}{\epsilon(\underline{k},\omega)}$$

The above two equations are known as fluctuation-dissipation theorems, because they relate a correlation of fluctuations (S_0 or I) to dissipation (proportional to $\mathrm{Im}\, m$ or $\mathrm{Im}\,\epsilon$). They can be proven true to all orders in the collisions.

B. In addition to the field correlation function, $I(\underline{k},\omega) = \frac{\langle|\underline{\mathcal{E}}(\underline{k},\omega)|^2\rangle}{VT}$, another correlation function of interest is the incoherent energy density. Consider the correlation function

$$W(\underline{r}-\underline{r}';t-t') = -\langle \underline{j}_0^f(\underline{r},t)\,\underline{\mathcal{E}}(\underline{r}',t')\rangle \tag{31}$$

in a stationary homogeneous plasma in the steady state. The work done by the spontaneous emission current on the fluctuating field is $W(0,0)$. Fourier transforming (31) yields,

$$W(\underline{k},\omega) = -\frac{\langle \underline{j}_0^f(\underline{k},\omega)\cdot\underline{\mathcal{E}}(\underline{k},\omega)^*\rangle}{VT} \tag{32}$$

Since
$$W(\underline{r},t) = \sum_{\underline{k},\omega}\frac{1}{VT}\,e^{i(\underline{k}\cdot\underline{r}-\omega t)}\,W(\underline{k},\omega)$$
$$\rightarrow \int\frac{d^3\underline{k}}{(2\pi)^3}\frac{d\omega}{2\pi}\,e^{i(\underline{k}\cdot\underline{r}-\omega t)}\,W(\underline{k},\omega),$$

it follows that

$$W(\underline{r}=0,t=0) = \int\frac{d^3\underline{k}}{(2\pi)^3}\frac{d\omega}{2\pi}\,W(\underline{k},\omega),$$

so that $W(\underline{k},\omega)$ is the work associated with an interval about $(\underline{k},\omega)$. Consider one of the integrals,

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} W(\underline{k},\omega) = \int_{0}^{\infty} \frac{d\omega}{2\pi}\left[W(\underline{k},\omega) + W(\underline{k},-\omega)\right]$$

Send $\underline{k} \longrightarrow -\underline{k}$ in the $d^3\underline{k}$ integration over $W(\underline{k},-\omega)$ and use the reality condition to write

$$W(\underline{r}=0,t=0) = \int \frac{d^3\underline{k}}{(2\pi)^3}\; W(\underline{k}), \tag{33}$$

$$W(\underline{k}) \equiv \int_0^{\infty} \frac{d\omega}{2\pi}\left[W(\underline{k},\omega) + W(\underline{k},\omega)^*\right] = \int_0^{\infty} \frac{d\omega}{\pi}\, \mathrm{Re}\, W(\underline{k},\omega), \tag{34}$$

we now only need consider positive frequencies, ω . Note $W(\underline{k})$ is the work associated with all modes having wave vector $\underline{k}$ (and the assumed polarization). We may evaluate $W(\underline{k},\omega)$, and then $W(\underline{k})$, using the result of (23). First, eliminating $\underline{j}_0^f$, gives us, in the vicinity of a particular resonance,

$$W(\underline{k})\Big|_{\omega\simeq\omega_r} = - \int_0^{\infty} \frac{d\omega}{\pi}\, \mathrm{Re}\left\{\frac{1}{-4\pi i}\,(\omega-\omega_r+i\gamma)\,\frac{\partial\, \mathrm{Re}\, m}{\partial\omega}\, I(\underline{k},\omega)\right\},$$

or,

$$W(\omega_r) = 2\,\frac{\partial}{\partial\omega}\,\mathrm{Re}\, m\Big|_{\omega=\omega_r}\; \gamma\; \underbrace{\int_{\omega_r} \frac{d\omega}{\pi}\; \frac{I(\underline{k},\omega)}{8\pi}}_{=\frac{I(\omega_r)}{8\pi}} \tag{34'}$$

In this steady state, lossy system, we may therefore identify the energy density associated with mode ω_ℓ by

$$\mathcal{U}(\omega_\ell) \equiv \frac{\partial \mathrm{Re}\, m}{\partial \omega} \frac{I(\omega_\ell)}{8\pi}, \tag{35}$$

since the energy loss rate is 2γ. The work, $W(\omega_\ell)$ may also be evaluated by eliminating $\mathcal{E}$ from equation (32), and this leads to an expression in terms of spontaneous emission alone. Alternatively, we may use equation (29) of the last lecture to write

$$2\gamma\, \mathcal{U}(\omega_\ell) = \frac{4\pi\; S_o(\omega_\ell)}{\left.\frac{\partial \mathrm{Re}\, m}{\partial \omega}\right|_{\omega = \omega_\ell}} \equiv W_o(\omega_\ell) \tag{36}$$

This is a form of Kirchoff's law. In equilibrium it is easy to prove that the energy density in a Langmuir wave with wave vector $\underline{k}$ is given by

$$\mathcal{U}(\omega_L) \cong k_B T \tag{37}$$

By integrating over all $\underline{k}$ with $k < k_D$, it is possible to show that the total energy density in Langmuir waves in equilibrium is proportional to $\frac{k_D^3}{n} n(k_B T)$, provided $k \ll k_D$. In thermodynamic equilibrium $\mathcal{U}(\omega_\ell) \approx \mathcal{U}(\underline{k})$, the total energy density of all modes with a common wave vector $\underline{k}$ ($\mathcal{U}(\underline{k})$ can be found exactly using Kramers-Kronig relations). In the absence of steady state we have the WKB energy conservation law:

$$\frac{\partial}{\partial t} \mathcal{U}(\omega_k) + \underline{\nabla} \cdot \left(\underline{v}_g\, \mathcal{U}(\omega_\ell)\right) + 2\gamma\, \mathcal{U}(\omega_\ell) = W_o(\omega_k), \tag{38}$$

where the group velocity, $\underline{v}_g \equiv \frac{\partial \omega_\ell}{\partial \underline{k}}$, and $\mathcal{U}(\omega_\ell)$ now has some weak space and time behaviour.

C. Next we return to the microscopic equations (1) and (2) and consider them in more detail, including considerations <u>beyond</u> the linear approximation. With part of the coherent field $\underline{A}$ split off and assumed to be part of the background plasma and indicated by $\underline{A}^o$ (we suppress superscripts s) we may write the equations as:

$$\mathcal{O}f = -\frac{\partial}{\partial \underline{v}} \cdot \left[\underline{A}' f + \langle \underline{a} \mathfrak{F} \rangle \right] \tag{39}$$

$$\mathcal{O}\mathfrak{F} = -\frac{\partial}{\partial \underline{v}} \cdot \left[\underline{a} f + \underline{A}' \mathfrak{F} + a \mathfrak{F} - \langle a \mathfrak{F} \rangle \right], \tag{40}$$

where, $\mathcal{O} = \frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{r}} + \underline{A}^o \cdot \frac{\partial}{\partial \underline{v}}$ is the (41)

"background" Vlasov operator, and $\underline{A}' = \underline{A} - \underline{A}^o$. First, we review the kind of perturbation theory we are considering here. Let us consider two separate problems. In the first, "coherent" situation one considers the propagation of finite-amplitude "test waves" in the plasma. For this purpose, equation (40) is either ignored, or used to construct an equilibrium (Boltzmann) binary-collisional model for $\langle \underline{a} \mathfrak{F} \rangle$ in terms of f . In the collisionless approximation, $\langle \underline{a} \mathfrak{F} \rangle$ is ignored. One must then solve the nonlinear equation

$$\mathcal{O}f = -\frac{\partial}{\partial \underline{v}} \cdot \underline{A}' f \tag{42}$$

for f in a power series in $\underline{E}'$ ($\underline{B}'$ can be eliminated through Maxwell's equation, (9)). Then $f = f_0 + f_1 + f_2$ in order of increasing powers of E'.

f_0 is the solution to

$\mathcal{O} f_0 = 0$, plus Maxwell's equations for $\underline{E}_o$ and B_o (43a)

f_1 (as a linear function of $\underline{E}'$) is the solution to

$$\mathcal{O} f_1 = -\frac{\partial}{\partial v} \cdot \underline{A}' f_0 \tag{43b}$$

and, in general f_n depends on f_{n-1} :

$$\mathcal{O} f_n = -\frac{\partial}{\partial \underline{v}} \cdot \underline{A}' f_{n-1} , \tag{43c}$$

When $\underline{E}_o$ and $\underline{B}_o$ are zero or independent of space and time, these equations may be solved by Fourier analysis. Formally, we may indicate solutions by

$$f_1 = -\mathcal{O}^{-1} \frac{\partial}{\partial \underline{v}} \cdot \underline{A}' f_0 , \tag{44a}$$

$$f_2 = (-1)^2 \mathcal{O}^{-1} \frac{\partial}{\partial \underline{v}} \cdot \left(\underline{A}' \Theta^{-1} \frac{\partial}{\partial \underline{v}} \cdot (A' f_0) \right) \tag{44b}$$

$$f_3 = (-1)^3 \mathcal{O}^{-1} \frac{\partial}{\partial \underline{v}} \cdot \left(\underline{A}' \Theta^{-1} \frac{\partial}{\partial \underline{v}} \cdot \underline{A}' \Theta^{-1} \frac{\partial}{\partial \underline{v}} \cdot \underline{A}' f_0 \right) \tag{44c}$$

Assuming Fourier analysis in a homogeneous, stationary plasma, we can calculate $f_n(\underline{v}, \underline{k}, \omega)$, and then the current $j^n(\underline{k},\omega) \equiv q \int d^3v\, \underline{v}\, f_n(\underline{v},\underline{k},\omega)$, of either species. The magnetic field is eliminated by Maxwell's equation (9). Following equation (15), we use the following notation for the linear and non-linear currents:

$$j_\ell^1(K) = -i\omega\, \chi_{\ell m}(K)\, E_m(K), \tag{45a}$$

$$j_\ell^{(2)}(K) = -i\omega \int dK_{12}\, \chi_{\ell m n}(K, K_1 K_2) E_m(K_1) E_n(K_2), \tag{45b}$$

$$j_\ell^{(3)}(K) = -i\omega \int dK_{123}\, \chi_{\ell m n p}(K, K_1 K_2 K_3) E_m(K_1) E_m(K_2) E_p(K_3), \tag{45c}$$

where $K = (\underline{k}, \omega)$, and

$$dK_{12} \equiv \frac{d^3\underline{k}_1}{(2\pi)^3}\,\frac{d\omega_1}{2\pi}\,\frac{d^3\underline{k}_2}{(2\pi)^3}\,\frac{d\omega_2}{2\pi}\,(2\pi)^4\delta(\omega-\omega_1-\omega_2)\,\delta^3(\underline{k}-\underline{k}_1-\underline{k}_2) \tag{46}$$

$$dK_{123} = \frac{d^3\underline{k}_1\,d\omega_1\,d^3\underline{k}_2\,d\omega_2\,d^3\underline{k}_3\,d\omega_3}{(2\pi)^{12}}\,(2\pi)^4\delta(\omega-\omega_1-\omega_2-\omega_3)\,\delta^3(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3) \tag{47}$$

We shall be concerned with a simple but powerful model for calculating the nonlinear susceptibility coefficients χ in case of a field-free background plasma. The expansion parameter in the above scheme is roughly the square root of the total coherent energy density associated with $\underline{E}$, over the total particle energy density ($n k_B T$ in equilibrium). We will return later to solutions of problems associated with $\underline{j}^{(2)}$ and $\underline{j}^{(3)}$ but for now we turn to an analogous problem involving the incoherent fields.

In this "incoherent" situation, one absorbs any important coherent fields into $\underline{A}^o$, assumes f can be found, and attempts to solve equations (39) and (40) for $\mathfrak{F}$ as a power series in ε :

$$\mathcal{O} f = -\frac{\partial}{\partial \underline{v}} \cdot \langle \underline{a}\, \mathfrak{F} \rangle \tag{48}$$

$$\mathcal{O} \mathfrak{F} = -\frac{\partial}{\partial \underline{v}} \cdot \left[\underline{a} f + \underline{a}\mathfrak{F} - \langle \underline{a}\mathfrak{F} \rangle \right] \tag{49}$$

The right side of equation (48) contains both collisional effects and effects of the incoherent wave spectrum reacting back on the coherent particle-distribution-function. Such effects are called "quasilinear". If we ignore collisional and quasilinear effects by setting the right side of (48) equal to zero, and assume $f = f_o$ is known, then we may expand $\mathfrak{F}$ in powers of ε : $\mathfrak{F} = \mathfrak{F}_o + \mathfrak{F}_1 + \mathfrak{F}_2 + \mathfrak{F}_3 + \ldots\ldots$

The analogues of equation (43) are then

$$\sigma \mathfrak{F}_0 = 0, \tag{50a}$$

the solution to which defines the Cerenkov-like processes, and

$$\sigma \mathfrak{F}_1 = -\frac{\partial}{\partial \underline{v}} \cdot \left[\underline{a} f_0 + a \mathfrak{F}_0 - \langle a \mathfrak{F}_0 \rangle \right]$$

Here, we may generally ignore the terms* $\underline{a}\,\mathfrak{F}_0 - \langle a \mathfrak{F}_0 \rangle$ as they govern higher order collisional processes, and are generally smaller than $\underline{a}\,f_0$. Thus, we have

$$\sigma \mathfrak{F}_1 = -\frac{\partial}{\partial \underline{v}} \cdot a f_0 \tag{50b}$$

$$\sigma \mathfrak{F}_2 = -\frac{\partial}{\partial \underline{v}} \left[\underline{a} \mathfrak{F}_1 - \langle a \mathfrak{F}_1 \rangle \right] \tag{50c}$$

$$\sigma \mathfrak{F}_3 = -\frac{\partial}{\partial \underline{v}} \left[\underline{a} \mathfrak{F}_2 - \langle \underline{a} \mathfrak{F}_2 \rangle \right] \tag{50d}$$

Several comments are now in order:

1. For an equilibrium plasma, the expansion of $\mathfrak{F}$ in powers of ε is essentially an expansion in the plasma parameter and therefore an expansion in collisions. This is because the total wave energy density (integrated up to $k = k_D$) is $\sim k_D^3/n\, n\,\theta$, where k_D^3/n is the inverse number of particles in a Debye sphere, often called the plasma parameter. This is easily proved from equation (37).

2. By inverting (50b) and solving for $\underline{j}^{f1}(K) \equiv \int d^3v\, \underline{v}\, \mathfrak{F}_1(\underline{v}, K)$, one arrives at the relation,

$$\underline{j}^{f1}(K) = \underline{\underline{\sigma}}(K) \cdot \underline{\varepsilon}(K), \tag{51a}$$

*Both terms must be ignored together, to insure $\langle \mathfrak{F}_1 \rangle = 0$.

which is the incoherent analogue of (11), and says that the fluctuating field $\underline{\mathcal{E}}$ produces a linear fluctuating polarization current density $\underline{j}^{fl}$ of the same magnitude (governed by the linear conductivity $\underline{\underline{\sigma}}$) that a coherent field would produce a coherent polarization current density. (See also equation (21)).

LECTURE III

June 11 , 1974

A. REVIEW: We have been comparing the linear and nonlinear constitutive relations for coherent and incoherent currents in terms of fields. Two separate cases are being studied:

1. Coherent case: All fluctuations and collisions are ignored. Known external coherent fields $\underline{E}_o$ and $\underline{B}_o$ are grouped with the Vlasov operator, $\mathcal{O} = \frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{r}} + \underline{A}_o \cdot \frac{\partial}{\partial \underline{v}}$, where $\underline{A}^o = \frac{q}{m}(\underline{E}_o + \frac{\underline{v}}{c} \times \underline{B}_o)$ is the coherent acceleration. Equation (42) for the distribution function f is solved by a perturbation expansion, $f = f_o + f_1 + f_2 + \ldots$ in the self-consistent fields (contained in the acceleration A'). The current is then formed by taking the velocity moment of f. When Maxwell's equation (9) is used to eliminate the self-consistent magnetic field, B', the result is $\underline{j}$ as a perturbation expansion in the self-consistent electric field $\underline{E}'$. If $\underline{A}^o$ is independent of space and time, Fourier analysis leads to the Fourier components for $\underline{j}$ given in equations (45). This defines the linear and nonlinear susceptibilities, $\chi_{\ell m}(K)$, $\chi_{\ell mn}(K\, K_1 K_2)$, etc.

2. Incoherent case: Coherent self-consistent fields are either ignored, or grouped with known external fields in the Vlasov operator $\mathcal{O}$. Quasilinear effects of fluctuations on the distribution function f are ignored, by setting the right side of equation (48) equal to zero. It is assumed that the Vlasov equation together with the coherent Maxwell equations can be solved for f to all orders in $\underline{A}^o$. Equation (49) for the fluctuating (incoherent) distribution function $\mathfrak{F}$ is then solved by a perturbation expansion in the incoherent fields in $\underline{a}$. The zero-order solution for F_o is given by

$$\mathfrak{F}_0(\underline{v},\underline{r},t) = \sum_i \delta^3(\underline{r}-\underline{r}_i(t))\delta^3(\underline{v}-\dot{\underline{r}}_i(t)) - \left\langle \sum_i \delta^3(\underline{r}-\underline{r}_i(t))\delta^3(\underline{v}-\dot{\underline{r}}_i(t)) \right\rangle ,$$

where the trajectories $\underline{r}_i(t)$ are the solutions to the equations of motion in the external fields $\underline{E}_0$ and $\underline{B}_0$:

$$\ddot{\underline{r}}_i(t) = \underline{A}_0(\underline{r}_i(t), \dot{\underline{r}}_i(t), t)$$

In first order, if the "collisional" terms $\underline{a}\,\mathfrak{F}_0 - \langle \underline{a}\,\mathfrak{F}_0 \rangle$ are ignored, one must solve equation (50b) for $\mathfrak{F}_1$. Then, $\mathfrak{F}_2$, $\mathfrak{F}_3$, etc. are found by iteration from equations (50c), (50d), etc. For a space and time independent acceleration $\underline{A}_0$, the "background plasma" is stationary and homogeneous, and the solutions to equations (50) may be integrated to construct the linear and nonlinear incoherent currents in terms of the incoherent field $\underline{\mathcal{E}}$. The linear incoherent current is given in (51a).

B. NONLINEAR INCOHERENT CURRENTS IN A STATIONARY, HOMOGENEOUS BACKGROUND PLASMA

By comparing equations (50) with equations (44), it is possible to formally relate the nonlinear incoherent currents $j^{fn}(K) \equiv \int d^3\underline{v}\, \underline{v}\, \mathfrak{F}_n(\underline{v},k)$ to the incoherent fields $\underline{\mathcal{E}}(K)$ through the same susceptibilities as in (45), but with modified combinations of fields and correlation functions:

$$j_\ell^{f2}(K) = -i\omega \int dK_{12}\, \chi_{\ell m}(KK_1K_2)\left[\mathcal{E}_m(K_1)\mathcal{E}_n(K_2) - \langle \mathcal{E}_m(K_1)\mathcal{E}_n(K_2)\rangle\right] \tag{51b}$$

$$j^{f3}_{\ell}(K) = -i\omega \int dK_{123}\, \chi_{\ell m n p}(KK_1K_2K_3)\Big[\mathcal{E}_m(K_1)\mathcal{E}_n(K_2)\mathcal{E}_p(K_3)$$

$$-\mathcal{E}_m(K_1)\langle\mathcal{E}_n(K_2)\mathcal{E}_p(K_3)\rangle - \langle\mathcal{E}_m(K_1)\mathcal{E}_n(K_2)\mathcal{E}_p(K_3)\rangle\Big] \qquad (51c)$$

Equations (51), together with Maxwell's equation, (4) for $\underline{\mathcal{E}}$ in terms of $\underline{j}^f$ can be used as a basis for a theory* of weak turbulence. Before proceeding with this task, however, it is useful to prove certain symmetry properties of the susceptibilities, and to develop coherent counterparts of some of the phenomena of weak turbulence.

C. SYMMETRY PROPERTIES OF THE NONLINEAR SUSCEPTIBILITIES

From equations (45b) and (45c) we may prove certain symmetry properties of the nonlinear susceptibilities. By interchanging the integration variables K_1 and K_2 and the matrix summation indices m and n, it immediately follows that

$$\chi_{\ell mn}(KK_1K_2) = \chi_{\ell nm}(KK_2K_1),$$

$$\begin{aligned}\chi_{\ell mnp}(KK_1K_2K_3) &= \chi_{\ell mpn}(KK_1K_3K_2)\\ &= \chi_{\ell nmp}(KK_2K_1K_3)\\ &= \chi_{\ell pnm}(KK_3K_2K_1)\end{aligned} \qquad (52)$$

---

*A complete theory also requires the quasilinear equation (48), which can make a contribution to j^{f3}.

It is more difficult to prove symmetries involving the variable K and index ℓ. The nonlinear current $\underline{j}^{NL}(\underline{r},t)$ is related to a nonlinear polarization vector $\underline{D}^{NL}(\underline{r},t)$, by

$$\underline{j}^{NL} = \frac{\partial}{\partial t}\underline{D}^{NL}$$

Consider the space and time average of $\underline{D}^{NL}\cdot\underline{E}$:

$$\begin{aligned}\overline{U} &\equiv \int_V \frac{d^3r}{V}\int_0^T \frac{dt}{T} D_\ell^{NL}(\underline{r},t)\,E_\ell(\underline{r},t) \\ &= \int dK\, D_\ell^{NL}(K)\,E_\ell(K)^*,\end{aligned} \tag{53}$$

where $dK \equiv d^3\underline{k}\,d\omega/(2\pi)^4$. Let

$$\begin{aligned}D_\ell^{NL}(K) = D_\ell^{(2)}(K) \equiv \int dK_1\, dK_2\, (2\pi)^4\,\delta(K-K_1-K_2) \\ \chi_{\ell mn}(K\,K_1,K_2)\,E_m(K_1)\,E_n(K_2)\end{aligned} \tag{54}$$

It is possible now to prove additional symmetry theorems. Since $\overline{U}$ is real we can take the complex conjugate, use the reality conditions $E_m(K_1)^* = E_m(-K_1)$, and $E_n(K_2)^* = E_n(-K_2)$, and then send K_1, K_2 into $-K_1$, $-K_2$ to prove that

$$\chi_{\ell mn}(-K,-K_1,-K_2) = \chi_{\ell mn}(K,K_1,K_2)^* \tag{55}$$

Also, by using $E_\ell(K)^* = E_\ell(-K)$, sending $K \to -K$, and interchanging K and K_1, etc, we can prove that

$$\chi_{\ell mn}(-K,K_1,K_2) = \chi_{nm\ell}(-K_2,K_1,K) = \chi_{m\ell n}(-K_1,K,K_2), \tag{56}$$

Combining (55) and (56), it follows that

$$\chi_{\ell mn}(k, k_1, k_2)^* = \chi_{nm\ell}(k_2, -k_1, k) \tag{57}$$

D. COHERENT NONLINEAR WAVES

The currents $\underline{j}^{(2)}$ and $\underline{j}^{(3)}$ of (45b) and (45c) can act as sources for waves in exactly the same way that $\underline{j}_{EXT}$ acts as a source in equation (12). By setting $\underline{j}_{EXT}$ equal to zero in equation (7) and setting $\underline{j} = \underline{j}_e + \underline{j}_i = \underline{j}^{(1)} + \underline{j}^{(2)} + \underline{j}^{(3)}$, we arrive at the equation

$$\begin{aligned} &\underline{\underline{M}}(K) \cdot \underline{E}(K) = -4\pi i\, \underline{j}^{NL}(K), \\ &\underline{j}^{NL}(K) = \underline{j}^{(2)}(K) + \underline{j}^{(3)}(K), \end{aligned} \tag{58}$$

for a homogeneous, stationary background plasma.

The solution of (58), together with (45) yields properties of non-linear coherent waves. A particularly important set of solutions can be found, using the so-called "parametric approximation".

E. "PARAMETRIC APPROXIMATION"

In the parametric approximation, the field $\underline{E}(\underline{r}, t)$ is taken to consist of a "pump", $\underline{E}^0(\underline{r}, t) = \bar{\underline{E}}^0 \cos(\underline{k}_0 \cdot \underline{r} - \omega_0 t)$, which is periodic, and assumed, known, and a part $\underline{E}'(\underline{r}, t)$, (which is smaller in amplitude) whose properties we seek. That is,

$$\underline{E}(\underline{r}, t) = \bar{\underline{E}}^0 \cos(\underline{k}_0 \cdot \underline{r} - \omega_0 t + \phi) + \underline{E}'(\underline{r}, t) \tag{59}$$

The object is to compute the effect of the field $\underline{E}^{0}$ on the normal modes of $\underline{E}'$ when $\underline{E}^{0}$ is still small enough to assure the convergence of the current expansions. We shall see that instabilities can be induced in $\underline{E}'$. The Fourier transform of (59) is

$$\underline{E}(K) = \frac{\bar{\underline{E}}_0}{2}(2\pi)^4\delta(K-K_0) + \frac{\bar{\underline{E}}_0^{*}}{2}(2\pi)^4\delta(K+K_0) + \underline{E}'(K) \tag{60}$$

where,

$$\underline{E}^{o} \equiv \bar{\underline{E}}^{o} e^{i\varphi}, \text{ and } \delta(K \pm K_0) = \delta^3(\underline{k}-\underline{k}_0)\delta(\omega-\omega_0) \tag{61}$$

Equation (60) may be inserted into equations (45) to obtain the linear and nonlinear current responses. If we are not interested in waves at frequency $n\omega_0$, n = 0, 1, 2, ..., and if we assume that $|E^{o}| \gg E^{(1)}$, then the dominant terms in the nonlinear currents are given by

$$\begin{aligned} \frac{j_{\ell}^{(2)}(K)}{-i\omega} = \; & \chi_{\ell mn}(K, K_0, K-K_0)\, E_m^{o}\, E_n'(K-K_0) \\ & \chi_{\ell mn}(K, -K_0, K+K_0)\, E_m^{o*}\, E_n'(K+K_0) \end{aligned} \tag{62}$$

$$\begin{aligned} \frac{j_{\ell}^{(3)}(K)}{-i\omega} = \; & 3\chi_{\ell mnp}(K, K_0, -K_0, K)\, E_m^{o}\, E_n^{o*}\, E_p'(K) \\ & + \frac{3}{2}\chi_{\ell mnp}(K, K_0, K_0, K-2K_0)\, E_m^{o}\, E_n^{o}\, E_p'(K) \\ & + \frac{3}{2}\chi_{\ell mnp}(K, -K_0, -K_0, K+2K_0)\, E_m^{o*}\, E_n^{o*}\, E_p'(K) \end{aligned} \tag{63}$$

We have used the symmetry conditions of equation (52) here to combine terms, In these terms the pump $\underline{E}^0$ beats against the self-consistent field $E'(K+n_1 K_0)$ to produce a nonlinear polarization current, $j^{NL}(K+n_2 K_0)$, where, n_1 and n_2 are integers. Fourier components separated by an integer multiple of the periodicity of the pump are thereby coupled. The nonlinear current, $j^{NL}(K+n_2 K_0)$ can act in equation (58) as a source for $E'(K+n_2 K_0)$. Thus, the $\chi_{\ell m n}$ are three-wave coupling coefficients, the $\chi_{\ell m n p}$ are four-wave coupling coefficients, and so forth.

1. Three-wave interactions (two coupled normal modes). Suppose we retain only the second-order current, $j^{(2)}$, and assume a diagonal matrix $\underline{\underline{M}}$. Then equation (54) becomes

$$\frac{\epsilon_\ell(K)}{-4\pi} E_\ell(K) = \chi_{\ell m n}(K, K_0, K-K_0) E^0_m E_m(K-K_0) + \chi_{\ell m m}(K, -K_0, K+K_0) E^{0*}_m E_m(K+K_0), \tag{64}$$

where

$$\epsilon_\ell(K) \equiv M_{\ell\ell}(K)/\omega = \frac{c^2 k_\ell^2}{\omega^2} - \frac{c^2 k^2}{\omega^2} + \epsilon_{\ell\ell}(K), \tag{65}$$

where there is no sum over ℓ on the left in (64). (The ℓ- index merely labels the diagonal element, $\epsilon_\ell(K)$). Similar equations can be generated from this, with $E_\ell(K \pm K_0)$ on the left side by displacing $K \to K \pm K_0$, etc. Continuing this procedure generates an infinite set of linear homogeneous equations for the mode amplitudes. Note, however, that $E_n(K)$ and $E_m(K-nK_0)$ can only be considered independent amplitudes if ω has a restricted domain, so that ω and $\omega - \omega_0$ take on different values. One way of accomplishing this is to restrict ω to lie in the internal $0 \leq \omega \leq \omega_0$. Then, for each n , $\omega - n\omega_0$ covers a different domain, and the totality of all n's covers the whole real ω - axis. The equations coupled to (64) and generated from it are:

$$\frac{\epsilon_\ell(K-K_0)}{-4\pi} E_\ell(K-K_0) = \chi_{\ell mn}(K-K_0, K_0, K-2K_0) E_m^0 E_n(K-2K_0)$$
$$+ \chi_{\ell mn}(K-K_0, -K_0, K) E_m^{0*} E_n(K) \tag{65}$$

$$\frac{\epsilon_\ell(K+K_0)}{-4\pi} E_\ell(K+K_0) = \chi_{\ell mn}(K+K_0, K_0, K) E_m^0 E_n(K)$$
$$+ \chi_{\ell mn}(K+K_0, -K_0, K+2K_0) E_m^{0*} E_n(K+2K_0), \tag{66}$$

and so forth. In general, contributions from $j^{(3)}$ and higher order currents must be included in this set.

Next, we assume that $\mathrm{Re}\,\omega$ is <u>near</u> a positive linear mode frequency, $\omega_a(\underline{k})$, associated with a polarization direction, $\hat{\underline{e}}^a$, and that $\mathrm{Re}\,\omega - \omega_0$ is near a negative* mode frequency, $-\omega_b(\underline{k}_0 - \underline{k})$ associated with a polarization direction, $\hat{e}^b$. For the 3-wave case, we also assume that $\mathrm{Re}\,\omega - n\omega_0$, for n equal to a positive or negative integer <u>other</u> than 0 and 1, is not near any normal mode frequency. More precisely, $\mathrm{Re}\,\omega - n\omega_0$ must be further away from any normal mode frequency ω_c than the associated damping rate, γ_c, of that mode. Under these conditions it is generally safe to make the approximation that, of all the Fourier components, $\underline{E}(K - nK_0)$, the largest amplitudes correspond to

$$E(K)\,\hat{\underline{e}}^a \qquad \text{and} \qquad E(K-K_0)\,\hat{\underline{e}}^b$$

This means we ignore the $K+K_0$ term in equation (64), the $K-2K_0$ term in equation (65), and ignore equation (66) completely. The homogeneous

---

*The reality conditions on $\epsilon_b(\underline{k},\omega)$ guarantee that if there is a positive frequency solution $\omega_b(\underline{k})$ to $\mathrm{Re}\,\epsilon_b = 0$, then there will also be a solution $-\omega_b(-\underline{k})$, which is guaranteed to be negative in an isotropic plasma.

equations (64) and (65) may then be written in matrix form as

$$\begin{pmatrix} \epsilon_a(K) \quad , & \chi(K, K-K_0) \\ \chi(K,K-K_0)^*, & \epsilon_b(K-K_0) \end{pmatrix} \begin{pmatrix} E(K) \\ E(K-K_0) \end{pmatrix} \tag{67}$$

Here we have defined the following two quantities:

ϵ_a or ϵ_b is a form of the dielectric function ϵ_ℓ , valid near ω_a or ω_b , where the index ℓ refers to the proper polarization:

$$\epsilon_a = \underline{\hat{e}}^a \cdot \underline{\underline{M}} \cdot \underline{\hat{e}}^a / \omega, \tag{68}$$

We also define,

$$\chi(K, K-K_0) = \hat{e}^a_\ell \chi_{\ell m n}(K, K_0, K-K_0) E^\circ_m \hat{e}^b_n \tag{69}$$

Since this susceptibility has two arguments K and $K-K_0$, it is not to be confused with the other linear and nonlinear susceptibilities. The arguments K and $K-K_0$ can be regarded as "matrix" indices, referring to which Fourier components $E(K-nK_0)$, (or normal modes) are coupled together. Finally, we have used the symmetry relation of equation (57) to express the off-diagonal matrix elements in (67) as complex conjugates of each other.

The new normal modes are then found by setting the determinant of the matrix in equation (67) equal to zero:

$$\epsilon_a(K)\, \epsilon_b(K-K_0) = |\chi(K, K-K_0)|^2 \tag{70}$$

If we make resonant approximations for ϵ_a and ϵ_b, in the forms

$$\epsilon_a(K) = \frac{\partial \operatorname{Re} \epsilon_a}{\partial \omega}\bigg|_{\omega=\omega_a(\underline{k})} (\omega - \omega_a(\underline{k}) + i\,\gamma_a(\underline{k})) \tag{71}$$

and

$$\epsilon_b(K-K_0) = \frac{\partial \operatorname{Re} \epsilon_b(K-K_0)}{\partial(\omega-\omega_0)}\bigg|_{\omega-\omega_0=-\omega_b(\underline{k}_0-\underline{k})} (\omega-\omega_0+\omega_b+i\,\gamma_b(\underline{k}_0-\underline{k})), \tag{72}$$

then the determinental condition (70) can be written as a quadratic equation in ω :

$$(\omega-\omega_a+i\,\gamma_a)(\omega-\omega_a+\Delta+i\,\gamma_b)+\Gamma^2=0 \tag{73}$$

The frequency mismatch Δ is defined as

$$\Delta \equiv \omega_a(\underline{k}) - \omega_b(\underline{k}_0-\underline{k}) - \omega_0 , \tag{74}$$

and the coupling constant Γ^2 by,

$$\Gamma^2 \equiv \frac{|\chi(K,K-K_0)|^2}{-\dfrac{\partial \operatorname{Re}\epsilon_b(K-K_0)}{\partial(\omega-\omega_0)}\bigg|_{\omega-\omega_0=-\omega_b(\underline{k}_0-\underline{k})} \dfrac{\partial \operatorname{Re}\epsilon_a(K)}{\partial\omega}\bigg|_{\omega=\omega_a(\underline{k})}} \tag{75}$$

We can find the conditions under which Γ^2 is positive as follows: For positive energy waves, we see from equation (35), that

$$0 < \frac{\partial \operatorname{Re} m_\ell(\underline{k},\omega)}{\partial\omega} = \operatorname{Re}\epsilon_\ell(\underline{k},\omega) + \omega\,\frac{\partial \operatorname{Re}\epsilon_\ell(\underline{k},\omega)}{\partial\omega} .$$

However, at a real normal mode frequency, $\omega_r(\underline{k})$, $\mathrm{Re}\,\epsilon_\ell$, vanishes. Thus, for positive energy waves,

$$\omega_r \left.\frac{\partial \mathrm{Re}\,\epsilon_\ell}{\partial \omega}\right|_{\omega=\omega_r} > 0, \tag{76}$$

and it follows that $\left.\frac{\partial \mathrm{Re}\,\epsilon_\ell}{\partial \omega}\right|_{\omega=\omega_r}$ has the same sign as ω_r.

Thus, as long as ω_a and $-\omega_b$ have opposite signs, Γ^2 is positive. However, if they have the same sign, Γ^2 is negative. We will refer to this later when commenting about the stability of down-shifted (Stokes) versus up-shifted (anti-Stokes) frequencies. For the present case, Γ^2 is positive.

One immediate solution to equation (73) is

$$\begin{aligned} \omega &= \omega_a \\ \Delta &= 0 \\ \Gamma^2 &= \gamma_a \gamma_b \end{aligned} \tag{77}$$

Since the new normal mode frequency is purely real whereas the old normal modes were damped, this value of Δ and Γ^2 corresponds to an instability threshold. The vanishing of Δ is a frequency selection rule (quantum mechanical energy conservation)

$$\omega_0 = \omega_a + \omega_b \tag{78}$$

The coupling constant, Γ^2, must be at least equal to $\gamma_a \gamma_b$, for the pump to overcome linear damping. The physical basis for this "parametric" instability is evident in equations (64) and (65).

The mode $E(K)$ beats with the pump to produce the mode $E(K-K_0)$. The mode $E(K-K_0)$ then beats again with the pump to reproduce $E(K)$. This "regeneration", in which $E(K)$ acts as a source for itself is one way of understanding three-wave parametric instabilities.

There are many kinds of three-wave parametric instabilities, even when there is no D.C. magnetic field. For an isotropic plasma, with electron temperature much longer than ion temperature, there are three linear normal modes,

$$\omega = \pm\omega_T(k) \equiv \pm\left(\omega_p^2 + c^2k^2\right)^{1/2} \qquad \text{(transverse electromagnetic)}$$

$$\omega = \pm\omega_L(k) \equiv \pm\left(\omega_p^2 + 3\,\Theta_e k^2/m_e\right)^{1/2} \qquad \text{(longitudinal electron plasma or "Langmuir" wave)}$$

$$\omega = \pm\omega_{ia}(k) \equiv \Theta_e k/m_i \qquad \text{(ion-acoustic wave)}$$

Some parametric instabilities which can be formed out of these waves are listed below, according to the identity of ω_a and ω_b . In each case the pump may be at ω_T or at ω_L , so there are at least 10 cases:

ω_a	ω_b	T Y P E
ω_{ia}	ω_L	electron - ion decay
ω_L	ω_L	degenerate $2\omega_p$
ω_{ia}	ω_T	stimulated Brillouin scattering
ω_L	ω_T	stimulated Raman scattering
ω_T	ω_T	degenerate electromagnetic

In addition to these, there are non-resonant or "quasimode" parametric instabilities, in which particles, rather than waves enter, in place of one of the frequencies ω_a or ω_b, and four-wave parametric instabilities, in which the mode $E(K+K_o)$ can be resonant. An example of the former is stimulated Compton scattering. Examples of the latter are self-focusing or oscillating two-stream instabilities. We shall return to these other cases later.

LECTURE IV

June 13 , 1974

A. CONTINUATION OF DISCUSSION OF 3-WAVE PARAMETRIC INSTATIBILIES

The growth rates of any of the three-wave parametric instabilities discussed in the last lecture can be obtained in terms of the frequency mismatch Δ and the coupling constant Γ^2 by solving the quadratic equation, (73) :

$$\omega = \omega_a - \frac{1}{2}\left[i(\gamma_a+\gamma_b)-\Delta\right] \pm \left(\frac{1}{4}\left[i(\gamma_a-\gamma_b)+\Delta\right]^2 - \Gamma^2\right)^{1/2}, \tag{79}$$

It is of more interest to insert a solution of form $\omega = \omega^{NL} - i\gamma^{NL}$ into (73) and solve for γ^{NL} and ω^{NL} (damping and frequency of the new normal mode). We find γ^{NL} obeys the equation

$$(\gamma_a - \gamma^{NL})(\gamma_b - \gamma^{NL})\left[1 + \frac{\Delta^2}{(\gamma_a+\gamma_b-2\gamma^{NL})^2}\right] = \Gamma^2 \tag{80}$$

The instability threshold condition, $\gamma^{NL} = 0$, therefore, gives Γ^2 as a function of Δ :

$$\Gamma^2_{th} = \gamma_a \gamma_b \left[1 + \frac{\Delta^2}{(\gamma_a + \gamma_b)^2}\right] \tag{81}$$

The minimum threshold is obviously at perfect frequency matching, $\Delta = 0$. Since Γ^2_{th} is proportional to $|E^0|^2$, the pump must overcome the linear damping to produce instability. For $\Gamma^2 \gtrsim \Gamma^2_{th}$ we can expand equation (80) about $\gamma^{NL} = 0$ to obtain the growth rate near threshold,

$$\gamma^{NL} = (\gamma_a+\gamma_b)\ \frac{1-\Gamma^2/\Gamma_{th}^2}{\dfrac{(\gamma_a+\gamma_b)^2}{\gamma_a\gamma_b} - \dfrac{4\Delta^2}{(\gamma_a+\gamma_b)^2+\Delta^2}} \tag{82}$$

The maximum growth rate slightly above threshold occurs for $\Delta = 0$, and is given by

$$\gamma^{NL}\Big|_{max} = \frac{\gamma_a\gamma_b - \Gamma^2}{\gamma_a+\gamma_b} \tag{83}$$

One can similarly find the real frequency ω^{NL} of the parametrically-coupled new normal mode. ω^{NL} will be near ω_a by assumption, which is a restriction on the allowed magnitude of Γ^2 or the pump field. This is no problem, since a pump large enough to shift a damping rate of magnitude γ_a into a growth rate of magnitude $\gtrsim \gamma_a$ results in a frequency shift on the order of γ_a/ω_a, which is small. In order to calculate frequency shifts consistently, it is necessary to take into account 4-wave interactions.

1. Other solutions to the 3-wave interaction problem

The new normal mode with negative damping rate γ^{NL} given in (82) is not the only new normal mode. The quadratic equation has two solutions, given by the plus and minus signs in equation (79). The other mode is more highly damped, as we can see for the case $\Delta = 0$, in which, from (79),

$$\omega = \omega_a - \frac{i}{2}(\gamma_a+\gamma_b) \pm i\left(\frac{1}{4}[\gamma_a-\gamma_b]^2 + \Gamma^2\right)^{1/2} \tag{84}$$

In addition to the two roots in (79), there are another two roots generated by these, and given by $\omega_0 - \omega^*$. The existence of these roots will be proven later from symmetry arguments. Note that the additional roots have the same growth or damping rates as the roots in (79), and their real frequency is near $\omega_0 - \omega_a$, which is near ω_b for small Δ. Thus, the new normal mode has a component near ω_a and a component near ω_b, both with the same growth rate !

2. Down-shifted (Stokes) versus up-shifted (anti-Stokes) waves

Since we have assumed ω_a and ω_b are positive, and the selection rule $\Delta \approx 0$ gives $\omega_b \approx \omega_0 - \omega_a$, ω_b is sometimes referred-to as a lower sideband of ω_0, or as a down-shifted wave, or as a Stokes line. If ω_a is negative, then $\omega_b \approx \omega_0 + |\omega_a|$ is an upper sideband or so-called anti-Stokes line. From equation (76) it then follows that $\left.\frac{\partial \mathrm{Re}\,\epsilon_a}{\partial \omega}\right|_{\omega=\omega_a}$ is negative, so Γ^2 is negative, by equation (75).

It is then clear from equation (80), that no threshold exists, and with $\Delta = 0$, equation (79) becomes

$$\omega = \omega_a - \frac{i}{2}\left(\gamma_a + \gamma_b\right) \pm \left(|\Gamma^2| - \frac{1}{4}\left(\gamma_a - \gamma_b\right)^2\right)^{1/2} \tag{85}$$

which has no unstable roots (for large Γ^2 there is merely a large frequency shift). Thus, one sometimes says that the three-wave parametric interaction de-stabilizes the Stokes line and stabilizes the anti-Stokes line. We shall refer to this later, when we treat turbulence analogues of 3-wave interactions.

B. LOW-FREQUENCY QUASI-MODE INSTABILITIES

There is a class of parametric instabilities, involving three coupled excitations, in which two are wave-like modes and the third involves single particle excitations, and sometimes is called a quasi-mode. The quasi-mode may be characterized by a complex frequency, but does not have the property that the damping rate is much less than the real part of the frequency. The example we shall treat here is the electron-ion decay instability, when the electron and ion temperatures are close or equal to each other. Under these conditions there is no "good" ion acoustic mode for which $|\gamma_{ia}| \ll |\omega_{ia}|$. This instability is, nevertheless, well-described by the three-wave dispersion relation of equation, (70), provided we do not make the resonant approximation for $\epsilon_b(K-K_o)$. Then, using (71) for ϵ_a, and assuming $\omega_a(k)$ is a linear frequency such as ω_L or ω_T, we obtain

$$\omega-\omega_a(k)+i\gamma_a(\underline{k})-\frac{|\chi(K,K-K_o)|^2}{\left.\frac{\partial \mathrm{Re}\,\epsilon}{\partial\omega}\right|_{\omega=\omega_a(\underline{k})}\epsilon_b(K-K_o)} \tag{86}$$

From the real part of this expression we can find $\mathrm{Re}\,\omega = \omega_a(k)+ \mathcal{O}(|E^o|^2)\approx\omega_a(k)$ for weak pumping. Then, assuming a parametric growth rate $|\gamma^{NL}|<\mathrm{Re}\,\omega$, we can set ω on the right side of (86) equal to ω_a. By then taking the imaginary part of (86), we arrive at the following expression for the total damping rate γ^{NL} of the new parametric normal mode:

$$\gamma^{NL}=\gamma_a(k)-\frac{|\chi(K,K-K_o)|^2}{\left.\frac{\partial \mathrm{Re}\,\epsilon}{\partial\omega}\right|_{\omega=\omega_a}}\,\mathrm{Im}\,\frac{1}{\epsilon_b(\underline{k}-\underline{k}_o,\omega_a(\underline{k})-\omega_o)}, \tag{87}$$

The condition for maximum growth can be determined from tables of the plasma dispersion function, from which the collisionless expression for ϵ at low frequencies can be found. We note that the pump-dependent part of the damping rate is proportional to $+\mathrm{Im}\,\epsilon_b(\omega_a(\underline{k})-\omega_o)$. From equation (19), assuming positive energy waves, it is easy to see that $\mathrm{Im}\,\epsilon_b$ is negative (growth) when $\omega_a-\omega_o$ is negative, and positive (damping) when $\omega_a-\omega_o$ is positive. Thus, when $\omega_a<\omega_o$, we have a destabilized Stokes line, ω_a , and when $\omega_a>\omega_o$, we have a stabilized anti-Stokes line, ω_a .

Equation (87) gives the correct growth rate of the new normal mode component near ω_a , and of the component near $\omega_o-\omega_a$ (quasi-mode component), when $\omega_o-\omega_a$ is a frequency determined by ion* particle dynamics. It also gives the correct result when $\omega_o-\omega_a$ is near a resonant wave frequency, $-\omega_b$. With the resonance approximation, (72), for ϵ_b , and, assuming $\Delta = 0$ and $\gamma_b \gg \gamma_a$, equation (87) reduces to exactly (83) for the maximum growth rate near threshold (we must be near threshold, so that frequency shifts are not important).

When ω_a is equal to the Langmuir frequency $\omega_L(k)=(\omega_p^2+3v_e^2k^2)^{1/2}$, equation (87) may be used to determine the growth rate for the equal-temperature parametric electron-ion decay instability. From tables of the plasma dispersion function it can be shown that $-\mathrm{Im}\,\epsilon_b$ is a maximum when

$$\omega_a-\omega_o=-1.7\,|k_o-k|\,\frac{\omega_{pi}}{k_{De}} \tag{88}$$

This is the effective frequency-matching condition for this case, and corresponds to $-\mathrm{Im}\,\epsilon_b = 0.58$. It turns out that this instability is fairly well treated by making a resonant approximation about the ion acoustic frequency for ϵ_b , and setting $\gamma_b/\omega_b \approx 1$.

---

*For electron quasi-mode instabilities, such as stimulated Compton scattering, it is necessary to include four-wave interactions.

C. SIMPLE MODELS FOR DETERMINING THE COUPLING CONSTANT, Γ

In linear wave theory, a cold or warm two-fluid model can predict many of the properties of waves, especially properties which do not depend on velocity-space interactions, such as the real resonant wave frequency. For these properties, it is not necessary to resort to a kinetic model such as the Vlasov or Boltzmann-like kinetic equation. The same situation is true, fortunately, for the calculation of the nonlinear susceptibilities. Simple fluid calculations of the nonlinear susceptibilities are almost always adequate for nonlinear wave and turbulence theories, although a kinetic description of the linear susceptibilities is usually necessary, either to include the correct damping, or to describe quasi-mode effects properly. The basis for a kinetic theory calculation of the nonlinear susceptibilities has been laid in the past few lectures, but in this section we will present some simple but powerful fluid models, for the case of no D.C. magnetic field.

1. Cold plasma theory of three coupled high-frequency waves

The nonlinear cold plasma equations are adequate only for computing the susceptibilities which couple three high-frequency waves (waves which linearly satisfy $\omega_R(k) \gg k V_s$, where $V_s = \Theta_s / m_s$, and Θ_s is the temperature of species s). In the absence of an external magnetic field, the model from which we can compute nonlinear currents consists of the following set of equations,

$$\left.\begin{aligned} &\left(\frac{\partial}{\partial t} + \underline{v}^s \cdot \nabla\right) \underline{v}^s = \frac{q_s}{m_s}\left(\underline{E} + \frac{\underline{v}^s}{c} \times \underline{B}\right), \quad \underline{\nabla} \times \underline{B} = -\frac{1}{c}\frac{\partial \underline{E}}{\partial t} \\ &\frac{\partial n^s}{\partial t} + \underline{\nabla} \cdot n^s \underline{v}^s = 0, \qquad \underline{j} = \sum_s q_s n^s \underline{v}^s \end{aligned}\right\} \qquad (89)$$

Henceforth, we shall omit the species superscript s. The Fourier transform of the momentum equation is

$$-i\omega \underline{v}(K) + i\int dK_{12}\, \underline{v}(K_1)\cdot \underline{k}_2 \underline{v}(K_2) = \frac{q}{m}\underline{E}(K) + \frac{q}{mc}\int dK_{12}\, \underline{v}(K_1)\times \underline{B}(K_2)$$

If we eliminate $\underline{B}$ in terms of $\underline{E}$ using the Fourier transform of Maxwell's equation, the result is

$$\underline{v}(K) = \frac{iq}{m\omega}\underline{E}(K) + \frac{iq}{m\omega}\int dK_{12}\frac{\underline{k}_2}{\omega_2}\, \underline{v}(K_1)\cdot\underline{E}(K_2) + \frac{1}{\omega}\int dK_{12}\, \underline{k}_2\cdot\underline{v}(K_1)\left[\underline{v}(K_2) + \frac{ie}{m\omega_2}\underline{E}(K_2)\right] \tag{90}$$

From equation (90), we can iterate to find the velocity in powers of $\underline{E}$ as $\underline{v} = \underline{v}^{(0)} + \underline{v}^{(1)} + \underline{v}^{(2)} + \ldots$. Assume $\underline{v}^{(0)} = 0$. Then $\underline{v}^{(1)}$ is obtained from the first term in (90):

$$\underline{v}^{(1)}(K) = \frac{iq}{m\omega}\underline{E}(K). \tag{91}$$

Substituting this velocity into the second and third terms in (90) generates $\underline{v}^{(2)}$. Since $\underline{v}^{(1)}$ goes as m^{-1}, we need only deal with the electrons when calculating the nonlinear susceptibilities. The second-order electron velocity is

$$\underline{v}^{(2)}(K) = -\frac{e^2}{m^2\omega}\int dK_{12}\,\frac{\underline{k}_2}{\omega_1\omega_2}\,\underline{E}(K_1)\cdot\underline{E}(K_2).$$

This contribution comes entirely from the second term in (90), which is part of the $\underline{v}^{(1)}\times\underline{B}$, or $\underline{E}\times\underline{B}$ term in the second-order momentum equation. The last term vanishes because part of $\underline{v}^{(1)}\times\underline{B}$ cancels against $\underline{v}^{(1)}\cdot\underline{\nabla}\,\underline{v}^{(1)}$. It is desireable to write $\underline{v}^{(2)}(K)$ in a symmetric form by interchanging integration variables K_1 and K_2. The result is

$$\underline{v}^{(2)}(K) = -\frac{e^2\underline{k}}{2m^2\omega}\int dK_{12}\,\frac{\underline{E}(K_1)\cdot\underline{E}(K_2)}{\omega_1\,\omega_2} \tag{92}$$

This velocity is polarized in the $\underline{k}$ direction, and therefore can only act as a source for longitudinal fields.

However, the nonlinear electron current is generated from

$$\underline{j}(K) = -e\int dK_{12}\, n(K_1)\,\underline{v}(K_2), \tag{93}$$

so in second order we have

$$\underline{j}^{(2)}(K) = -e\int dK_{12}\left[n^{(0)}(K_1)\,\underline{v}^{(2)}(K_2) + n^{(1)}(K_1)\,\underline{v}^{(1)}(K_1)\right]$$

Using the continuity equation, this reduces to

$$\underline{j}^{(2)}(K) = -e\,n_0\,\underline{v}^{(2)}(K) - e\,n_0\int dK_{12}\,\frac{\underline{k}_1\cdot\underline{v}^{(1)}(K_1)\,\underline{v}^{(1)}(K_2)}{\omega_1}, \tag{94}$$

Symmetrizing the second term, we arrive at

$$\underline{j}^{(2)}(K) = \frac{e^3 n_0}{m^2}\int\frac{dK_{12}}{\omega_1\,\omega_2}\left[\frac{\underline{k}\cdot\underline{E}(K_1)\cdot\underline{E}(K_2)}{2\omega} + \frac{\underline{k}_1\cdot\underline{E}(K_1)\underline{E}(K_2)}{2\omega_1} + \frac{\underline{k}_2\cdot\underline{E}(K_2)\underline{E}(K_1)}{2\omega_2}\right] \tag{95}$$

Comparing this expression with the definition of the nonlinear susceptibility, χ_{lmn} in (45b), we arrive at the result,

$$\chi_{lmn}(K K_1 K_2) = \frac{e^3 n_0 i}{2m^2\omega\omega_1\omega_2}\left[\frac{k_l\delta_{mn}}{\omega} + \frac{k_{1m}\delta_{ln}}{\omega_1} + \frac{k_{2n}\delta_{lm}}{\omega_2}\right] \tag{96}$$

This is only a valid model for 3-wave interactions or instabilities in which all three waves are high frequency, such as the degenerate $2\omega_p$ parametric instability and stimulated Raman scattering.

The reason is that the second-order current depends on products of first-order velocities. In this model, the first-order velocities arise from a balance between the inertial and electric-field force terms in the momentum equation. However, at low frequencies the inertial term becomes less important than the pressure term, which we have omitted. With an isothermal equation of state, the plasma can be in equilibrium with the self-consistent field at low frequencies. This leads to a large first-order density response at long wavelengths which can only be predicted from a model which includes pressure. Interactions between two high and one low frequency wave generally have stronger coupling coefficients $\chi_{\ell mn}$ for this reason and lead to more important interactions and instabilities. We therefore next treat a warm fluid model which includes an isothermal pressure and leads to a more general expression for $\chi_{\ell mn}$ than equation (96).

2. <u>Warm fluid theory of nonlinear susceptibilities</u>

The momentum equation now contains an additional pressure term, $\gamma \Theta_e \underline{\nabla} n / m n$ on the left side. It is, therefore, convenient to introduce the variable

$$\ell \equiv \ln n/n_0$$

in place of the density. The pressure term is then $\gamma V_e^2 \underline{\nabla} \ell$, where γ is the ratio of specific heats, and $V_e^2 \equiv \Theta_e/m_e$. The momentum equation is coupled to the continuity equation through this term, and we can write the continuity equation in terms of the variable ℓ as

$$\frac{\partial \ell}{\partial t} + \underline{\nabla} \cdot \underline{V} + \underline{V} \cdot \underline{\nabla} \ell = 0$$

The variable ℓ can then be eliminated from the momentum equation by taking a time-derivative and using the continuity equation. The result is

$$\left[\frac{\partial^2}{\partial t^2} - \gamma v_e^2 \underline{\nabla}\,\underline{\nabla}\cdot\right] \underline{v} = -\frac{e}{m}\frac{\partial}{\partial t}\underline{E}$$

$$-\frac{\partial}{\partial t}(\underline{v}\cdot\underline{\nabla}\,\underline{v}) - \underline{\nabla}\,\underline{v}\cdot(\underline{v}\cdot\underline{\nabla}\,\underline{v}) - \frac{e}{m}\underline{\nabla}\,\underline{v}\cdot\underline{E}$$

$$-\frac{e}{mc}\frac{\partial}{\partial t}(\underline{v}\times\underline{B}) - \underline{\nabla}\,\underline{v}\cdot\frac{\partial}{\partial t}\underline{v}$$

Fourier transforming this equation, and eliminating the magnetic field in favor of $\underline{E}$, as before, we arrive at the following equation:

$$\begin{aligned}\underline{\underline{T}}^{-1}(K)\cdot\underline{v}(K) &= \underline{E}(K) + \int dK_{12}\left(\frac{\underline{k}_2}{\omega_2} - \frac{\underline{k}}{\omega}\right)\underline{v}(K_1)\cdot\underline{E}(K_2) \\ &+ \frac{m_i}{e}\int dK_{12}\,\underline{v}(K_1)\cdot\underline{k}_2\left[\underline{v}(K_2) + \frac{ei}{m\omega_2}\underline{E}(K_2)\right] \\ &+ \frac{m_i}{e\omega}\int dK_{12}\,\underline{k}\,\underline{v}(K_1)\cdot\underline{v}(K_2)\,\omega_2 \\ &+ \frac{m_i\underline{k}}{e\omega}\int dK_{123}\,\underline{v}(K_1)\cdot\underline{v}(K_3)\,\underline{k}_3\cdot\underline{v}(K_2)\end{aligned} \qquad (97)$$

In this equation, the second order tensor $\underline{\underline{T}}^{-1}(K)$ is defined by

$$\underline{\underline{T}}^{-1}(K) \equiv \frac{im\omega}{e}\left(\underline{\underline{1}} - \frac{\gamma v_e^2\,\underline{k}\,\underline{k}}{\omega^2}\right)$$

Its inverse is the symmetric tensor

$$\underline{\underline{T}}(K) \equiv -\frac{ei}{m\omega}\left(\underline{\underline{1}} + \frac{\underline{k}\,\underline{k}\,\gamma v_e^2}{\omega^2 - \gamma v_e^2 k^2}\right), \qquad (98)$$

Operating with $\underline{\underline{T}}$ on both sides of (97) gives a form for the non-linear velocity $\underline{V}$ which is suitable for iteration to find $V^{(2)}$ and higher-order velocities in terms of $\underline{E}$. The first-order velocity is

$$\underline{V}^{(1)}(K) = \underline{\underline{T}}(K) \cdot \underline{E}(K) \tag{99}$$

From this one can form the first-order current and conclude that the $\underline{\underline{T}}$ matrix is essentially the linear electron conductivity in the warm fluid electron model:

$$\underline{\underline{\sigma}}(K) \equiv -e n_0 \underline{\underline{T}}(K) \tag{100}$$

After combining terms and symmetrizing in K_1 and K_2, we arrive at the following expression for the second-order velocity,

$$\underline{V}^{(2)}(K) = \frac{\underline{k} \cdot \underline{\underline{T}}(K)}{2} \int dK_{12} \left\{ \frac{2mi}{e} \left[\underline{\underline{T}}(K_1) \cdot \underline{E}(K_1) \right] \cdot \left[\underline{\underline{T}}(K_2) \cdot \underline{E}(K_2) \right] - \frac{1}{\omega} \underline{E}(K_1) \cdot \left[\underline{\underline{T}}(K_1) + \underline{\underline{T}}(K_2) \right] \cdot \underline{E}(K_2) \right\} \tag{101}$$

Finally, note that

$$\underline{k} \cdot \underline{\underline{T}}(K) = -\frac{ei}{m\omega} \frac{\omega^2 \underline{k}}{\omega^2 - \gamma V_e^2 k^2}, \tag{102}$$

so $V^{(2)}$ is longitudinal. The complete second-order current is obtained from (94) and is given by

$$\underline{j}^{(2)}(k) = -\frac{en_0}{2}\int dk_{12}\left\{\frac{im}{e}\underline{k}\cdot\underline{\underline{T}}(k)\underline{E}_1\cdot\left[2\underline{\underline{T}}_1\cdot\underline{\underline{T}}_2 + \frac{ei}{m\omega}\left(\underline{\underline{T}}_1+\underline{\underline{T}}_2\right)\right]\cdot\underline{E}_2 + \underline{\underline{T}}_1\cdot\underline{E}_1\left[\frac{\underline{k}_2}{\omega_2}\cdot\underline{\underline{T}}_2\cdot\underline{E}_2\right] + \underline{\underline{T}}_2\cdot\underline{E}_2\left[\frac{\underline{k}_1}{\omega_1}\cdot\underline{\underline{T}}_1\cdot\underline{E}_1\right]\right\}, \tag{103}$$

where we have abbreviated $\underline{\underline{T}}(k_1)$ by $\underline{\underline{T}}_1$, $\underline{E}(k_1)$ by $\underline{E}_1$, and so forth. We note immediately, that, in the limit of zero pressure $(v_e \rightarrow 0)$, $T_{ij} = -\frac{ei}{m\omega}\delta_{ij}$, and this expression reduces to (95). By again comparing with (45b) we arrive at the following general expression for the nonlinear susceptibility $\chi_{\ell mn}$:

$$\chi_{\ell mn}(kk_1k_2) = \frac{e^3 n_0 i}{2m^2\omega\omega_1\omega_2}\left\{\frac{k_i U_{i\ell}(k)}{\omega^2}\left[2\omega U_{mp}(k_1)U_{pn}(k_2) - \omega_2 U_{mn}(k_1) - \omega_1 U_{mn}(k_2)\right] + \frac{k_{1p}U_{pm}(k_1)U_{\ell n}(k_2)}{\omega_1} + \frac{k_{2p}U_{pn}(k_2)U_{m\ell}(k_1)}{\omega_2}\right\} \tag{104}$$

where,

$$U_{ij}(k) \equiv \frac{im\omega}{e}T_{ij}(k) = \delta_{ij} + \frac{k_i k_j \gamma v_e^2}{\omega^2 - \gamma v_e^2 k^2} \tag{105}$$

We note that with a more general definition of $\underline{\underline{U}}$, the expression (103) for $\chi_{\ell mn}$ can be generalized still further. For example, if $\underline{\underline{\sigma}}(k)$ is the linear conductivity of a warm plasma in a <u>static magnetic field</u>, then the expression (104) is still valid, provided $\underline{\underline{U}}$ is given by (see (100)),

$$\underline{\underline{U}}(K) \equiv -\frac{im\omega}{e^2 n_0}\,\underline{\underline{\sigma}}(K) \tag{106}$$

To our knowledge, this is the first time that the general formula (104) has ever been written down. It gives the correct three-wave coupling for all parametric instabilities in an isotropic homogeneous plasma, and is probably correct for most cases with a magnetic field (in some magnetic cases, a similar ion contribution must be added in).

When dealing with scalar fields we are concerned with the quantity

$$\begin{aligned} \hat{e}_\ell\, \chi_{\ell mn}(KK_1K_2)\,\hat{e}^1_m\,\hat{e}^2_n = \frac{e^3 n_0 i}{2m^2\omega\omega_1\omega_2} \Bigg\{ & \frac{\underline{k}\cdot\underline{\underline{U}}(k)\cdot\underline{\hat{e}}}{\omega^2}\Big[2\omega\,\underline{\hat{e}}^1\cdot\underline{\underline{U}}_1\cdot\underline{\underline{U}}_2\cdot\underline{\hat{e}}^2 \\ & -\omega_2\,\underline{\hat{e}}^1\cdot\underline{\underline{U}}_1\cdot\underline{\hat{e}}^2 - \omega_1\,\underline{\hat{e}}^1\cdot\underline{\underline{U}}_2\cdot\underline{\hat{e}}^2\Big] + \frac{\underline{k}_1\cdot\underline{\underline{U}}_1\cdot\underline{\hat{e}}^1\,\underline{\hat{e}}\cdot\underline{\underline{U}}_2\cdot\underline{\hat{e}}^2}{\omega_1} \\ & + \frac{\underline{k}_2\cdot\underline{\underline{U}}_2\cdot\underline{\hat{e}}^2\,\underline{\hat{e}}\cdot\underline{\underline{U}}_1\cdot\underline{\hat{e}}^1}{\omega_2}\Bigg\} \equiv \chi(KK_1K_2) \end{aligned} \tag{107}$$

LECTURE V

June 18 , 1974

A. CONTINUED DISCUSSION OF WARM FLUID THEORY OF NONLINEAR SUSCEPTIBILITIES

Last time, we found a general expression from warm fluid theory for the contracted 3-wave nonlinear susceptibility,

$$\chi(K K_1 K_2) = \hat{e}_\ell \, \chi_{\ell mn}(K K_1 K_2) \, \hat{e}^1_m \, \hat{e}^2_n \tag{108}$$

Since $\chi(K K_1 K_2)$ always appears in integrals over dK_{12}, we may take $\omega = \omega_1 + \omega_2$, in all cases. We are interested in cases, for which two waves are high-frequency and one is either high or low. If we arbitrarilly choose $|\omega_1|$ and $|\omega_2|$ to be high frequency, then $|\omega|$ can be high or low frequency, and the other permutations can be generated from the symmetry relations, (52), and (57). Then $\underline{\underline{U}}_1$ and $\underline{\underline{U}}_2$ are $\approx \underline{\underline{1}}$, and (107) reduces to

$$\chi(K K_1 K_2) \Big|_{\substack{|\omega_1| \gg v_e k_1 \\ |\omega_2| \gg v_e k_2}} = \frac{i e^3 n_0}{2 m^2 \omega \omega_1 \omega_2} \left\{ \frac{\underline{k} \cdot \hat{\underline{e}} \, \hat{\underline{e}}^1 \cdot \hat{\underline{e}}^2}{\omega} \, \frac{1}{1 - \gamma v_e^2 k^2/\omega^2} + \frac{\underline{k}_1 \cdot \hat{\underline{e}}^1 \hat{\underline{e}} \cdot \hat{\underline{e}}^2}{\omega_1} + \frac{\underline{k}_2 \cdot \hat{\underline{e}}^2 \hat{\underline{e}} \cdot \hat{\underline{e}}^1}{\omega_2} \right. \tag{109}$$

This is rather simple generalization of the result from cold fluid theory, equation (96). The sole difference is the factor $(1 - \gamma v_e^2 k^2/\omega^2)^{-1}$ which multiplies the first term. When $\omega^2 \gg \gamma v_e^2 k^2$, this factor is essentially one. When $\omega^2 \ll \gamma v_e^2 k^2$, it is equal to $\omega^2/\gamma v_e^2 k^2 \ll 1$. However, this factor is always multiplied by ω^{-2}, so the net factor when $\omega = \omega_{HF}$ is ω_{HF}^{-2} , whereas, the net factor when $\omega = \omega_{LF}$ is $(\gamma v_e^2 k^2)^{-1}$. The ratio of the coupling of two high-frequency waves and one low-frequency wave to the coupling of three high frequency waves is therefore given by

$$\frac{\chi(K^{LF}, K_1^{HF}, K_2^{HF})}{\chi(K^{HF}, K_1^{HF}, K_2^{HF})} = \frac{\omega_{HF}^2}{\gamma v_e^2 k^2} \gg 1 . \tag{110}$$

We conclude that interactions which involve two high-frequency, and one low-frequency wave are dominant over interactions involving three high-frequency waves. The reason for this is that the linear low-frequency susceptibility is $4\pi\chi(K) = k_D^2/k^2$, whereas the high-frequency susceptibility is $4\pi\chi(K) = -\omega_p^2/\omega_{HF}^2$. First-order low-frequency density responses to a given field can, therefore, be much larger than similar high-frequency density responses. The origin of the enhancement factor for the ordering we have chosen is the factor $\underline{k}\cdot\underline{\underline{T}}(K)\cdot\hat{e}$ which is proportional to $\chi(K)$, the low-frequency density response. This factor comes from $\underline{V}^2(K)$ (see equation (101)), which, therefore, depends indirectly on the low-frequency first-order density response. Such dependence could have been seen more directly, if we had chosen ω_2 as the low-frequency wave. The dominant contribution to $\chi(K^{HF}, K_1^{HF}, K_2^{LF})$ would then have come from the part of the nonlinear current proportional to $n^{(1)}(K_2)\underline{v}^{(1)}(K_1)$, and would, by the symmetry relations, have resulted in exactly the same enhancement as (110).

From equation (109), together with the symmetry relations (52) and (57), and the definitions (69) for $\chi(K, K-K_0)$, and (75) for Γ^2, we can find the coupling constants Γ^2 for all of the three-wave parametric instabilities. We list some of these below, in terms of the directed velocity, $\underline{V}_0$ of electrons in the pump field, $\underline{E}_0$

$$\underline{V}_0 \equiv \frac{e\underline{E}_0}{m_e\omega_0}$$

Instability	Coupling constant, Γ^2
Electron-ion decay instability (equal or unequal electron and ion temperatures)	$\frac{(\underline{k}\cdot\underline{v}_0)^2}{16}\ \frac{\omega_{ia}(k)}{\omega_0}\ \frac{k_{De}^2}{k^2}$
Degenerate $2\omega_{pe}$ instability	$\frac{(\hat{\underline{k}}\cdot\underline{v}_0)^2}{4}\ \frac{\omega_{pe}^2}{\omega_0^2}\ (\hat{k}\cdot\underline{k}_0)^2$
Stimulated Brillouin scattering	$\frac{v_0^2}{16 v_e^2}\ (\hat{e}^0\cdot\hat{e}^T)^2\ \frac{\omega_{pe}\,\omega_{ia}(k_a)}{\omega_0^2}$,
Stimulated Raman scattering	$\frac{k_a^2 v_0^2}{16}\ (\hat{e}^0\cdot\hat{e}^T)^2\ \frac{\omega_{pe}}{\omega_T}$

B. THEORY OF 3-WAVE TURBULENCE

We now return to the incoherent model described at the beginning of Lecture III in which the only coherent fields present are stationary and homogeneous, assumed known, and retained in the background Vlasov operator, $\mathcal{O}$. The solution of $\mathcal{O}f_0 = 0$ for the coherent distribution function f_0 is also assumed known. Equation (51b), then relates the fluctuating fields $\underline{\mathcal{E}}(K)$ to the second-order fluctuating current $\underline{j}^{f2}$ by susceptibilities such as those we have just calculated. $\underline{j}^{f2}$ must then be added as a source in Maxwell's equation (21) for the fluctuating field, $\underline{\mathcal{E}}(K)$. For uncoupled linear waves (diagonal problem), it assumes the form

$$m(K)\,\underline{\mathcal{E}}(K) = -4\pi i\left(\underline{j}_0^{\,f}(K) + \underline{j}^{f2}(K)\right). \tag{111}$$

If we take the scalar product of both sides of this equation with $\underline{\mathcal{E}}(K')$, and integrate over dK' , the result is,

$$m(K)\,I(K) = +\,4\pi i\left[W^{o}(K) + W^{(2)}(K)\right], \tag{112}$$

where,

$$W^{o}(K) \equiv -\,\frac{\langle \underline{j}_0^{f}(K)\cdot\underline{\mathcal{E}}^{*}(K)\rangle}{VT}, \tag{113}$$

$$W^{(2)}(K) \equiv -\int dK' \,\langle \underline{j}^{f2}(K)\cdot\underline{\mathcal{E}}(K')\rangle, \tag{114}$$

Here we have used the result (24) for correlations of fluctuations in a linear, homogeneous steady-state plasma, and have used the definition (26) of the spectral function, $I(K)$. $W^{(0)}$ and $W^{(2)}$ represent the work done by spontaneous emission and nonlinear currents on the fluctuating field $\mathcal{E}$. With $W^{(2)}$ neglected, we can generate from (112) the linear fluctuation-dissipation theorem, or Kirchoff's law, as in Lecture II. $W^{(2)}$ contains 3-mode weak turbulence effects which are analogous to the 3-wave parametric instabilities just considered. By substituting from the constitutive relation (51b) we obtain from (114),

$$W^{(2)}(K) = +\,i\omega\int dK'\int dK_{12}\;\chi'(K K_1 K_2)\,\langle\mathcal{E}(K')\mathcal{E}(K_1)\mathcal{E}(K_2)\rangle, \tag{115}$$

where

$$\chi'(K K_1 K_2) = \hat{e}_{\ell}^{K'}\,\chi_{\ell mn}(K K_1 K_2)\,\hat{e}^{1}_{m}\,\hat{e}^{2}_{n},$$

and the $\mathcal{E}$'s are now scalars. To reduce (115) further requires the evaluation of a triple field correlation function, $\langle\mathcal{E}(K')\mathcal{E}(K_1)\mathcal{E}(K_2)\rangle$.

To do this, we must first understand how nonlinearities produce correlations. Let us rewrite equation (111) as

$$\mathcal{E}(K) = \mathcal{E}^{(0)}(K) - \frac{4\pi\omega}{m(K)} \int dK_{12}\, \chi(KK_1K_2) \Big[\mathcal{E}(K_1)\mathcal{E}(K_2) - \langle \mathcal{E}(K_1)\mathcal{E}(K_2)\rangle\Big], \tag{116}$$

where,

$$\mathcal{E}^{\circ}(K) \equiv -4\pi i\, j_0^f(k)/m(K), \tag{117}$$

and $\chi(KK_1K_2)$ differs from $\chi'(KK_1K_2)$ only by having the unit vector $\hat{e}_\ell^K$ in place of $\hat{e}_\ell^{K'}$. We can regard the nonlinear term in (116) as a source of "higher-order correlations". Suppose we attempt to solve (116) by iteration in $\mathcal{E}^{\circ}(K)$. The first iteration gives,

$$\mathcal{E}^1(K) = -\frac{4\pi\omega}{m(K)} \int dK_{12}\, \chi(KK_1K_2)\Big[\mathcal{E}^{\circ}(K_1)\mathcal{E}^{\circ}(K_2) - \langle \mathcal{E}^{\circ}(K_1)\mathcal{E}^{\circ}(K_2)\rangle\Big]. \tag{118}$$

We, likewise, expand the triple field correlation function as

$$\begin{aligned}\langle \mathcal{E}(K')\mathcal{E}(K_1)\mathcal{E}(K_2)\rangle \approx\ & \langle \mathcal{E}^{\circ}(K')\mathcal{E}^{\circ}(K_1)\mathcal{E}^{\circ}(K_2)\rangle \\ & + \langle \mathcal{E}^{1}(K')\mathcal{E}^{\circ}(K_1)\mathcal{E}^{\circ}(K_2)\rangle + \langle \mathcal{E}^{\circ}(K')\mathcal{E}^{1}(K_1)\mathcal{E}^{\circ}(K_2)\rangle \\ & + \langle \mathcal{E}^{\circ}(K')\mathcal{E}^{\circ}(K_1)\mathcal{E}^{1}(K_2)\rangle .\end{aligned} \tag{119}$$

We now make the random phase approximation, which states that,

$$\langle \mathcal{E}_1^{\circ}\, \mathcal{E}_2^{\circ} \rangle = -I(K_1)\,(2\pi)^4\, \delta(K_1 + K_2), \tag{120}$$

$$\langle \mathcal{E}_1^\circ \mathcal{E}_2^\circ \mathcal{E}_3^\circ \rangle = 0, \tag{121}$$

$$\langle \mathcal{E}_1^\circ \mathcal{E}_2^\circ \mathcal{E}_3^\circ \mathcal{E}_4^\circ \rangle = \langle \mathcal{E}_1^\circ \mathcal{E}_2^\circ \rangle \langle \mathcal{E}_3^\circ \mathcal{E}_4^\circ \rangle + \langle \mathcal{E}_1^\circ \mathcal{E}_3^\circ \rangle \langle \mathcal{E}_2^\circ \mathcal{E}_4^\circ \rangle + \langle \mathcal{E}_1^\circ \mathcal{E}_4^\circ \rangle \langle \mathcal{E}_2^\circ \mathcal{E}_3^\circ \rangle . \tag{122}$$

Equation (120) is familiar from linear homogeneous theory. The minus sign in equation (120) is associated with the fact that we have been using a scalar electric field $\mathcal{E}(K)$ which is related to the vector field $\underset{\sim}{\mathcal{E}}(K)$ by the unit vector $\hat{\underline{e}}(K)$. That is, $\underset{\sim}{\mathcal{E}}(K) = \hat{\underline{e}}(K)\, \mathcal{E}(K)$. One way to satisfy the reality condition $\underset{\sim}{\mathcal{E}}(-K) = \underset{\sim}{\mathcal{E}}(K)^*$ is to take

$$\mathcal{E}(-K) = -\,\mathcal{E}(K)^*,$$

$$\hat{\underline{e}}(-K) = -\,\hat{\underline{e}}(K)^*.$$

The second equation implies, for real polarizations, that

$$\hat{\underline{e}}(-K) = -\hat{\underline{e}}(K)$$

This is always true for longitudinal fields, and defines coordinate systems always relative to $\underset{\sim}{k}$ for transverse fields. We thus note that $\underset{\sim}{\mathcal{E}}(K)\cdot\underset{\sim}{\mathcal{E}}(K)^* = \mathcal{E}(K)\,\mathcal{E}(K)^* = -\mathcal{E}(K)\,\mathcal{E}(-K)$. This is the source of the minus sign in equation (120). Finally, we must understand tacitly that only two fields of the same polarization are to be considered as correlated in equation (120).

The reason equations (120) - (122) are called the random phase approximation can be understood by considering phase averages, instead of ensemble averages. Suppose the complex fluctuating field is written as an amplitude times a phase factor:

$$\underline{\mathcal{E}}^{\circ}(K) = \underline{A}(K)\, e^{i\varphi(K)} \tag{123}$$

where $\mathrm{Re}\,\underline{\mathcal{E}}^{\circ} = \underline{A}\cos\varphi$, and $\mathrm{Im}\,\mathcal{E}^{\circ} = \underline{A}\sin\varphi$.

The reality condition $\underline{\mathcal{E}}^{\circ}(-K) = \underline{\mathcal{E}}^{\circ}(K)^{*}$ then tells us that

$$\left.\begin{aligned} \underline{A}(-K) &= \underline{A}(K) \\ \text{and}\quad \varphi(-K) &= -\varphi(K). \end{aligned}\right\} \tag{124}$$

We define the phase average as an integral over all phase functions φ, which obey (124). Thus

$$\overline{\underline{\mathcal{E}}^{\circ}(K)} \equiv \int_{-\pi}^{+\pi} \frac{d\varphi}{2\pi}\, \underline{A}(K)\, e^{i\varphi} = 0,$$

In the ensemble of systems with different phases (all obeying $\varphi(-K) = -\varphi(K)$), the average of $e^{i\varphi}$ vanishes. Hence $\underline{\mathcal{E}}^{\circ}(K)$ is fluctuating. The correlation function,

$$\overline{\underline{\mathcal{E}}^{\circ}(K_1)\,\underline{\mathcal{E}}^{\circ}(K_2)} = \int_{-\pi}^{+\pi} \frac{d\varphi}{2\pi}\, \underline{A}(K_1)\,\underline{A}(K_2)\, e^{i(\varphi(K_1)+\varphi(K_2))}$$

also vanishes, unless $K_2 = -K_1$, so that $\varphi(K_1) + \varphi(-K_1) = 0$. This corresponds to equation (120). Similarly, $\overline{\underline{\mathcal{E}}^{\circ}(K_1)\underline{\mathcal{E}}^{\circ}(K_2)\underline{\mathcal{E}}^{\circ}(K_3)} = 0$, because there are no values of K_1, K_2 and K_3 for which $\varphi(K_1) + \varphi(K_2) + \varphi(K_3) = 0$. The correlation function $\overline{\underline{\mathcal{E}}^{\circ}(K_1)\underline{\mathcal{E}}^{\circ}(K_2)\underline{\mathcal{E}}^{\circ}(K_3)\underline{\mathcal{E}}^{\circ}(K_4)}$ involves an integrand with phase factor $\varphi_{TOT} = \varphi(K_1) + \varphi(K_2) + \varphi(K_3) + \varphi(K_4)$. There are several ways φ_{TOT} can vanish: Either $K_1 = -K_2$, and $K_3 = -K_4$; or $K_1 = -K_3$ and $K_2 = -K_4$; or $K_1 = -K_4$ and $K_2 = -K_3$. These correspond to the break-up of the four-field correlation function into products of two-field correlation functions in equation (122). To evaluate $W^{(2)}(K)$, we first use the expansion (119) for $\langle \mathcal{E}(K')\,\mathcal{E}(K_1)\,\mathcal{E}(K_2)\rangle$, followed by equation (118) for $\mathcal{E}'(K')$, $\mathcal{E}'(K_1)$, and $\mathcal{E}'(K_2)$.

The result is

$$W^{(2)}(K) = -4\pi i\omega \int dK' dK_1 dK_2 (2\pi)^4 \delta(K-K_1-K_2)\, \chi'(KK_1K_2) \cdot$$

$$\left[\frac{2\omega_1}{m(K_1)} \int dK_3\, dK_4 (2\pi)^4 \delta(K_1-K_3-K_4)\, \chi(K_1K_3K_4) \cdot \right.$$

$$\left\{ \langle \varepsilon^\circ(K')\, \varepsilon^\circ(K_2)\, \varepsilon^\circ(K_3)\, \varepsilon^\circ(K_4) \rangle - \langle \varepsilon^\circ(K')\varepsilon^\circ(K_2)\rangle \langle \varepsilon^\circ(K_3)\varepsilon^\circ(K_4)\rangle \right\}$$

$$+ \frac{\omega'}{m(K')} \int dK_3\, dK_4\, (2\pi)^4\, \delta(K'-K_3-K_4)\, \chi(K'K_3K_4)$$

$$\left. \left\{ \langle \varepsilon^\circ(K_1)\, \varepsilon^\circ(K_2)\, \varepsilon^\circ(K_3)\, \varepsilon^\circ(K_4) \rangle - \langle \varepsilon^\circ(K_1)\varepsilon^\circ(K_2)\rangle \langle \varepsilon^\circ(K_3)\varepsilon^\circ(K_4)\rangle \right\} \right]$$

We have also used here the symmetry in K_1 and K_2 to combine the $\varepsilon^1(K_2)$ term with the $\varepsilon^1(K_1)$ term. If we next use the random phase approximation, the first term in brackets, $\{\ \}$, becomes

$$\left\{ I(K')\, I(K_2)(2\pi)^8 \left[\delta(K'+K_3)\delta(K_2+K_4) + \delta(K'+K_4)\delta(K_2+K_3) \right] \right\}$$

and the second term in brackets, $\{\ \}$, becomes

$$\left\{ I(K_1)\, I(K_2)(2\pi)^8 \left[\delta(K_1+K_3)\delta(K_2+K_4) + \delta(K_1+K_4)\delta(K_2+K_3) \right] \right\}$$

Consequently,

$$\delta(K_1-K_3-K_4) = \delta(K+K') \quad \text{and} \quad \delta(K'-K_3-K_4) = \delta(K+K') .$$

The integrals over K' , K_3 and K_4 can immediately be performed, and yield

$$W^2(K) = +4\pi i\omega \int dK_{12}\, \mathcal{X}(KK_1K_2)\Big[\frac{2\omega_1}{m(K_1)}\Big\{\mathcal{X}(K_1,K,-K_2)$$
$$+\mathcal{X}(K_1,-K_2,K)\Big\}I(K)I(K_2) - \frac{\omega}{m(-K)}\Big\{\mathcal{X}(-K,-K_1,-K_2)$$
$$+\mathcal{X}(-K,-K_2,-K_1)\Big\}I(K_1)I(K_2)\Big]$$

Here, we have used the obvious symmetry $I(-K) = I(K)$, and also made use of $\delta(K+K')$ and the relation $\hat{e}(-K) = -\hat{e}(K)$ to write $\mathcal{X}'(KK_1K_2) = -\mathcal{X}(KK_1K_2)$.

If we now use the symmetry relation (52) for $\mathcal{X}$, then equation (112) for the steady-state spectral function becomes

$$m(K)\,I(K) = 4\pi i\, W^0(K)$$
$$-\omega I(K)\,4(4\pi)^2\int dK_{12}\,\frac{\mathcal{X}(KK_1K_2)\,\mathcal{X}(K_1,+K,-K_2)\,I(K_2)}{m(K_1)/\omega_1} \qquad (125)$$
$$+\frac{\omega^2 2(4\pi)^2}{m(K)}\int dK_{12}\,\mathcal{X}(KK_1K_2)\,\mathcal{X}(-K,-K_1,-K_2)\,I(K_1)I(K_2).$$

Finally, if we apply the symmetry relations (55) and (57), we find

$$-\mathcal{X}(K_1,K,-K_2) = \mathcal{X}^*(KK_1K_2) = \mathcal{X}(-K,-K_1,-K_2),$$

so

$$
\begin{aligned}
m(K)I(K) &= 4\pi i\, W^{\circ}(K) \\
&+ I(K)\, 4(4\pi)^2 \omega \int dK_{12} \frac{|\chi(KK_1K_2)|^2 I(K_2)}{m(K_1)/\omega_1} \\
&- \frac{2(4\pi)^2\omega^2}{m(-K)} \int dK_{12}\, |\chi(KK_1K_2)|^2 I(K_1) I(K_2)\ .
\end{aligned}
\tag{126}
$$

It is useful to write this in the form,

$$
\tilde{m}(K)\, I(K) = 4\pi i\, \tilde{W}(K), \tag{127}
$$

where the renormalized resonance function $\tilde{m}(K)$ is,

$$
\tilde{m}(K) = m(K) - 4(4\pi)^2 \omega \int dK_{12} \frac{|\chi(KK_1K_2)|^2 I(K_2)}{m(K_1)/\omega_1}, \tag{128}
$$

and the renormalized spontaneous emission $\tilde{W}^{\circ}$

$$
4\pi i\, \tilde{W}^{\circ}(K) = 4\pi i\, W^{\circ}(K) - \frac{2(4\pi)^2}{m(+K)^*}\omega^2 \int dK_{12}\, |\chi(KK_1K_2)|^2 I(K_1) I(K_2). \tag{129}
$$

Since we are now dealing with longitudinal turbulence, we may use dielectric functions $\epsilon = m/\omega$ to rewrite (128):

$$
\tilde{\epsilon}(K) = \epsilon(K) - 4(4\pi)^2 \int dK_{12} \frac{|\chi(K,K_1K_2)|^2 I(K_2)}{\epsilon(K_1)}\ . \tag{130}
$$

The turbulent alteration of the mode K is obtained by the condition, $\tilde{\epsilon}(K) = 0$ for a new normal mode, $\tilde{\omega}_a(k) - i\tilde{\gamma}_a(k)$, near some linear normal mode $Re\, \omega \approx \omega_a(k)$ of $\epsilon(K)$. Making resonant approximations, we can obtain the new frequency and damping rate. The new frequency is the solution to the following equation,

$$\mathrm{Re}\ \tilde{\epsilon}(k) = 0 = \left.\frac{\partial \mathrm{Re}\,\epsilon}{\partial\omega}\right|_{\omega=\omega_a} (\tilde{\omega}_a - \omega_a) - \tag{131}$$

$$4(4\pi)^2 \int dk_{12}\, |\chi(k k_1 k_2)|^2\, I(k_2)\, \mathrm{Re}\, \frac{1}{\epsilon(\tilde{\omega}_a - \omega_2; \underline{k} - \underline{k}_2)}$$

The new damping rate is determined by

$$\tilde{\gamma}_a(k) = \gamma_a(\underline{k}) - \frac{4(4\pi)^2 \int dk_{12}\, |\chi|^2 I(k_2)\, \mathrm{Im}\, \dfrac{1}{\epsilon(\tilde{\omega}_a - \omega_2; \underline{k} - \underline{k}_2)}}{\left.\dfrac{\partial \mathrm{Re}\,\tilde{\epsilon}}{\partial\omega}\right|_{\omega=\tilde{\omega}_a}} \tag{132}$$

For real frequencies $\tilde{\omega}_a$ that are not shifted too far from their linear values, ω_a , we can approximate $\tilde{\omega}_a \simeq \omega_a$ on the right sides of (130) and (131). The condition $\tilde{\epsilon} \simeq 0$ in (130) gives the turbulent analogue of equation (70), which can be written as

$$\epsilon_a(k) = \frac{(4\pi)^2\, |\chi(k, k_0, k - k_0)|^2\, |E_0|^2}{\epsilon_b(k - k_0)}\ . \tag{133}$$

The turbulent damping rate given in (132) is analogous to (82) for the coherent case, with an integral over the spectrum playing the role of pump. We shall return to this analogy later, but first we must study the consistency of our iteration scheme.

LECTURE VI

June 20, 1974

A. REVIEW OF THEORY OF 3-WAVE TURBULENCE

We have started with Maxwell's equation for the scalar fluctuating field $\mathcal{E}$ with sources up to and including the fluctuating polarization current produced by the beating of two fluctuating fields. In the diagonal approximation, this takes the form,

$$\mathcal{M}(K)\,\mathcal{E}(K) = -4\pi i j_0^f(K) - 4\pi\omega\int dK_{12}\,\chi(K\,K_1K_2)\left[\mathcal{E}_1\mathcal{E}_2 - \langle\mathcal{E}_1\mathcal{E}_2\rangle\right] \qquad (134)$$

By then multiplying this by $\mathcal{E}(K')$, integrating over K', and taking the ensemble average, this becomes

$$\mathcal{M}(K)\frac{\langle|\mathcal{E}(K)|^2\rangle}{VT} = -4\pi i\frac{\langle j_0^f(K)\,\mathcal{E}^*(K)\rangle}{VT} + 4\pi\omega\int dK'\int dK_{12}\,\chi(K\,K_1K_2)\,\langle\mathcal{E}(K')\,\mathcal{E}(K_1)\,\mathcal{E}(K_2)\rangle \qquad (135)$$

Our procedure for reducing (135) to a form which depended only on the spectral function $I(K) \equiv \langle|\mathcal{E}(K)|^2\rangle/VT$ was then to iterate from (134) in each of the fields in the triple-field correlation function in (135) and then to reduce four-field correlations to products of intensities by the random phase approximation, equations (120) - (122).

This led us to equation (127), for the spectrum in terms of itself. If we use the result $\mathcal{E}^*(K) = 4\pi i\, j_0^f(K)^*/\mathcal{M}(K)^*$ from the linear theory, then equation (127) - (129) may be put in the form

$$I(K) = \frac{(4\pi)^2\, \tilde{S}(K)}{\tilde{m}(K)\, m(K)^*} \tag{136}$$

where, $\tilde{m}(K)$ is the renormalized resonance function given by equation (128),

$$\tilde{m}(K) = m(K) - 4\,(4\pi)^2 \omega \int dK_{12} \frac{|\chi(KK_1K_2)|^2 I(K_2)}{m(K_1)/\omega_1}, \tag{128}$$

and $\tilde{S}$ is a <u>renormalized spontaneous emission</u>, given by

$$\tilde{S}(K) = S_0(K) - 2\omega^2 \int dK_{12} |\chi(KK_1K_2)|^2 I(K_1) I(K_2) \tag{137}$$

We recall from equation (28), that $S_0(K) = \langle |j_0^f(K)|^2 \rangle / VT$ is linear spontaneous emission, such as Cerenkov emission. The zeroes of $\tilde{m}(K)$ then gave us the turbulently renormalized new normal modes, which were, in fact, seen to bear a striking formal similarity to coherent 3-wave parametric instability theory. In fact, the iterations from (134) into (135) are analogous to the regeneration in the coherent parametric instability theory. However, before proceeding we must note a problem with the iteration scheme which has led to an asymmetric form of (136) with respect to the m's in the denominator. We shall see that the iteration scheme we have used is <u>not</u> a self-consistent scheme. Some selective summation is necessary to make a completely,self-consistent renormalization.

B. SELF-CONSISTENT RENORMALIZATION

The iteration scheme we have used starts with an assumed form for the spectrum which follows from equation (117):

$$I(K) = (4\pi)^2 \frac{S_0(K)}{|m(K)|^2}$$

Since this is just the linear result, it clearly is inconsistent with (136). One step towards self-consistency is to not assume any ordering of terms in (134), but simply to use the entire right side when iterating for each of the three fields in the triple correlation function in (135). If triple correlations such as $\langle j_0^f \mathcal{E}\mathcal{E} \rangle$ are ignored (random phase approximation), this leads to exactly the same result (equation 136) as before. However, the form of the spectrum now derived by using equation (134) in both $\mathcal{E}(K)$ and in $\mathcal{E}(K)^*$ when forming $\langle \mathcal{E}(K)\,\mathcal{E}(K)^* \rangle$ is easily shown to be

$$I(K) = (4\pi)^2 \frac{\tilde{S}(K)}{|m(K)|^2}, \tag{138}$$

where $\tilde{S}(K)$ is precisely the renormalized spontaneous emission given in (129). However, (138) is still inconsistent with (136), because of the denominators. To make a self-consistent scheme we must first rewrite (134) in the form,

$$\left[m(K) + m^{NL}(K)\right] \mathcal{E}(K) = -4\pi i\, j_0^f(K) + m^{NL}(K)\,\mathcal{E}(K)$$

$$-4\pi\omega \int dK_{12}\, \chi(KK_1K_2)\left[\mathcal{E}_1\mathcal{E}_2 - \langle \mathcal{E}_1\mathcal{E}_2 \rangle\right],$$

where m^{NL} is a turbulent correction to $m(K)$ which must be determined. If we then divide through by $m + m^{NL}$, the result is

$$\mathcal{E}(K) = \frac{1}{\tilde{m}(K)}\Big[-4\pi i j_0^f(K) + m^{NL}(K)\,\mathcal{E}(K)$$

$$-4\pi\omega\int dK_{12}\,\chi(KK_1K_2)\left[\mathcal{E}_1\mathcal{E}_2 - \langle\mathcal{E}_1\mathcal{E}_2\rangle\right]\Big], \tag{139}$$

where $\tilde{m}$ is <u>defined</u> as

$$\tilde{m}(K) = m(K) + m^{NL}(K) \tag{140}$$

If equation (139) is then used to iterate for each of the three fields in the triple correlation function on the right side of (135), and if the random phase approximation is then applied (triple correlations are ignored, and quadruple correlations are factored into products of double correlations), the result is

$$I(K) = (4\pi)^2\,\frac{\tilde{S}(K)}{|\tilde{m}(K)|^2}\;, \tag{141}$$

where $\tilde{m}(K)$ is defined implicitly by,

$$\tilde{m}(K) = m(K) - 4\,(4\pi)^2\omega\int dK_{12}\,\frac{|\chi(KK_1K_2)|^2 I(K_2)}{\tilde{m}(K_1)/\omega_1}\;. \tag{142}$$

The integrand in (142) has $\tilde{m}(K_1)$ in the denominator, whereas, the integrand in (128) has $m(K_1)$ in the denominator. In many cases, the small renormalizations in $\tilde{m}(K_1)$ make no practical difference. A resonant expansion is often made, so integrals over $\mathrm{Im}\,\tilde{m}(K_1)^{-1}$ are essentially independent of $\tilde{\gamma}$. Thus, the turbulent damping rate given in equation (132) of the last lecture is essentially unchanged.

Equation (141) is obviously more symmetric than (136), but we must now ask if it is consistent with the spectral function obtained by substituting from (139) for both $\varepsilon(K)$ and $\varepsilon(K)^*$ in the expression, $I(K) \equiv \langle \varepsilon(K)\varepsilon(K)^* \rangle / VT$. When this is done and the random phase approximation is applied, the result is

$$I(K) = \frac{(4\pi)^2 \tilde{S}(K) + |m^{NL}|^2 I(K)}{|\tilde{m}(K)|^2}, \tag{143}$$

where $\tilde{S}(K)$ is again defined precisely as in equation (137). Since $|m^{NL}|^2$ second order in I, the term $|m^{NL}|^2 I$ is third order, and thus can be ignored relative to $\tilde{S}(K)$ in this approximation. In this sense, (143) is consistent with (141) and we now have a self-consistent iteration scheme, based on equation (139). We note that each iteration is proportional to $\tilde{m}^{-1}$, which contains all orders in I, so a selective summation has been performed here, as in secularity-free perturbation theory. The self-consistent model of 3-wave turbulence now consists of equations (137) for $\tilde{S}(K)$, (142) for $\tilde{m}(K)$, and (141) for $I(K)$.

C. RESONANT APPROXIMATIONS AND KIRCHOFF'S LAW FOR WEAK TURBULENCE

Suppose we are interested in the turbulence which arise when some linear wave of frequency ω_a has a linear damping rate γ_a which is negative, due to some linear instability. Then a weak level of turbulence can create a new damping rate $\tilde{\gamma}_a$, which is positive. This turbulent dissipation is balanced by turbulent spontaneous emission in the manner described by equation (131). From this, we can derive a form of Kirchoff's law which is valid in the weakly turbulent plasma. Suppose we make a resonant approximation for $\tilde{m}(K)$ about the new normal mode with frequency $\tilde{\omega}_a$ and damping rate $\tilde{\gamma}_a$. Then (131) may be written as

$$I(\underline{k},\omega) = \frac{(4\pi)^2\,\tilde{S}_o(\underline{k},\omega)}{\left(\frac{\partial Re\,\tilde{m}}{\partial\omega}\Big|_{\omega=\tilde{\omega}_a}\right)^2\left[(\omega-\tilde{\omega}_a(k))^2+\tilde{\gamma}_a^2(k)\right]}$$

Assuming $\tilde{S}_o$ does not change appreciably over a frequency change on the order of $\tilde{\gamma}_a$, we may make a delta-function approximation, and write this as

$$I(\underline{k},\omega) = \frac{(4\pi)^2\,\tilde{S}_o(\underline{k},\tilde{\omega}_a)}{\left(\frac{\partial Re\,\tilde{m}}{\partial\omega}\Big|_{\omega=\tilde{\omega}_a}\right)^2\tilde{\gamma}_a(k)}\;\pi\,\delta(\omega-\tilde{\omega}_a(k)), \tag{144}$$

As in the linear case described at the end of the first lecture, we can then define the spectral function associated with a particular mode $\tilde{\omega}_a$ (near ω_a). In an isotropic plasma we always have modes at $\pm\tilde{\omega}_a(k)$ for a given $\underline{k}$, as in the linear theory. It is customary in the isotropic plasma to associate one spectral function with both of these signs, by writing

$$I(\underline{k},\omega) = \frac{(4\pi)^2\,\tilde{S}_0(\underline{k},\tilde{\omega}_a)}{\left(\frac{\partial Re\,\tilde{m}}{\partial\omega}\Big|_{\omega=\tilde{\omega}_a}\right)^2\tilde{\gamma}_a(k)}\;\pi\left[\delta(\omega-\tilde{\omega}_a(k))+\delta(\omega+\tilde{\omega}_a(k))\right] \tag{145}$$

We have used here the symmetry $\tilde{S}_0(-\tilde{\omega}_a)=\tilde{S}_0(\tilde{\omega}_a)$, which is easily proven in the isotropic plasma. The spectral function associated with $\pm\tilde{\omega}_a(\underline{k})$ is then given, as in (29), by

$$\begin{aligned}\underline{I}(\tilde{\omega}_a) &\equiv \int_{\text{resonance at }\pm\tilde{\omega}_a}\frac{d\omega}{2\pi}\,I(\underline{k},\omega)\\ &= \frac{(4\pi)^2}{\tilde{\gamma}_a\left(\frac{\partial Re\,\tilde{m}}{\partial\omega}\Big|_{\omega=\tilde{\omega}_a}\right)^2}\;\tilde{S}(\underline{k},\tilde{\omega}_a(k)),\end{aligned} \tag{146}$$

One may then parallel the linear derivation of Kirchoff's law which started from equation (23). We now start, however, from the nonlinear analogue of equation (23), namely equation (139), with $m^{NL}\mathcal{E}$ ignored on the right. If we proceed as in the linear theory (equations (31) - (36)), we arrive at the following turbulent Kirchoff's law:

$$2\tilde{\gamma}_a \tilde{U}(\omega_a) = \frac{4\pi\, \tilde{S}(\tilde{\omega}_a)}{\left.\frac{\partial \mathrm{Re}\, \tilde{m}}{\partial \omega}\right|_{\omega=\omega_a}} \equiv \tilde{W}(\omega_a), \tag{147}$$

where the turbulent energy density is defined as

$$\tilde{U}(\omega_a) = \left.\frac{\partial \mathrm{Re}\, \tilde{m}}{\partial \omega}\right|_{\omega=\tilde{\omega}_a} \frac{I(\tilde{\omega}_a)}{8\pi}, \tag{148}$$

in analogy with (35). For weak turbulence, the factor $\left.\frac{\partial \mathrm{Re}\, \tilde{m}}{\partial \omega}\right|_{\omega=\tilde{\omega}_a}$ is often approximately equal to the linear value, $\left.\frac{\partial \mathrm{Re}\, m}{\partial \omega}\right|_{\omega=\omega_a}$.

Equation (147) is, of course, the mathematical expression of the fact that turbulent spontaneous emission is balanced by turbulent damping. We remark that $\tilde{\gamma}_a = \gamma_a + \gamma_a^{NL}$, where γ_a is the linear damping rate, and γ_a^{NL} is proportional to an integral over I. Although γ_a may be large and negative, $\tilde{\gamma}_a$ is always small and positive when 3-wave turbulence is the dominant saturation effect. In practice, the steady state can often be thought of as arising from the balance of γ_a^{NL} against $-\gamma_a$, with spontaneous emission playing a negligible role. The spectrum can then be determined by solving the equation $\gamma_a^{NL} + \gamma_a = 0$. The levels of turbulent energy density then obtained from (148) can be much larger than the electron temperature Θ_e, but cannot be so large as to make the integrated energy over all $\underline{k}$ as large as $n\Theta_e$. Turbulence theory breaks down here. Before proceeding with solutions to $\gamma_a^{NL} + \gamma_a = 0$, we show the connection between the formalism we have developed and semiclassical "Golden-rule" methods of time dependent perturbation theory.

D. SEMICLASSICAL "GOLDEN-RULES"

We can make the connection with the semiclassical "Golden-rule" methods of time-dependent perturbation theory, as discussed by Tsytovich, for 3-wave interactions as follows:

Suppose we have a situation in which the real frequencies ω_a, ω_b and ω_c are not shifted very much by their mutual interaction. Suppose, further, that all three waves are "good"* electrostatic normal modes, but that ω_a and ω_c are high frequency, and ω_b is low frequency (since this is where we expect the largest coupling). From equation (132), the nonlinear contribution to the damping γ_a of wave "a" is given by,

$$\gamma_a^{NL}(\underline{k}) = \frac{-(8\pi)^2}{\left.\frac{\partial \mathrm{Re}\,\epsilon}{\partial \omega}\right|_{\omega=\omega_a(k)}} \int dK_1\, dK_2\, (2\pi)^4 \delta(\omega_a-\omega_1-\omega_2)\,\delta^3(\underline{k}-\underline{k}_1-\underline{k}_2)\,|\chi|^2\, I(K_2)\, \mathrm{Im}\frac{1}{\epsilon(K_1)} \tag{149}$$

In this expression, we have ignored the nonlinear shifts in ω_a, and assumed $\mathrm{Re}\,\tilde{\epsilon}$ is close to $\mathrm{Re}\,\epsilon$. Assume for simplicity that the high frequency turbulence is dominant, so that we can write

$$I(K_2) \approx I_c(k_2)\,\pi\left[\delta(\omega_2-\omega_c(k_2))+\delta(\omega_2+\omega_c)\right], \tag{150}$$

*In the sense that their real frequencies are much larger than their damping.

where we have used $I_c \equiv I(\omega_c)$, defined as in (145) and (146). We will ignore, for the time being, contributions to γ_a^{NL} from $I(\omega_b)$, which is a low-frequency turbulent contribution. Inserting (150) into (149), and performing the ω_2 integration, we obtain,

$$\gamma_a^{NL}(\underline{k}) = \frac{-(8\pi)^2\pi}{\left.\frac{\partial Re\,\epsilon}{\partial\omega}\right|_{\omega=\omega_a}} \int dK_1 \frac{d^3k_2}{(2\pi)^3}(2\pi)^4\delta^3(\underline{k}-\underline{k}_1-\underline{k}_2)|\chi|^2 I_c(\underline{k}_2)$$

$$\mathrm{Im}\frac{1}{\epsilon(K_1)}\left[\delta(\omega_a-\omega_1-\omega_c)+\delta(\omega_a-\omega_1+\omega_c)\right] \tag{151}$$

There are two neighborhoods of the integration variable ω_1 where a low frequency resonant response of $\mathrm{Im}\,\epsilon(K_1)^{-1}$ is possible. One is near $\omega_1 = +\omega_b$, and the other is near $\omega_1 = -\omega_b$. We can allow for both of these, by using the resonant approximation,

$$\mathrm{Im}\frac{1}{\epsilon(K_1)} = \frac{\pi}{\left.\frac{\partial Re\,\epsilon}{\partial\omega_1}\right|_{\omega_1=\omega_b}}\left[-\delta(\omega_1-\omega_b(k_1))+\delta(\omega_1+\omega_b(k_1))\right] \tag{152}$$

We have explicitly used the symmetry,

$$\left.\frac{\partial Re\,\epsilon}{\partial\omega_1}\right|_{\omega_1=\omega_b} = -\left.\frac{\partial Re\,\epsilon}{\partial\omega_1}\right|_{\omega_1=\omega_b}$$

If we now perform the ω_1 integration in (151), assuming ω_a, ω_b, and ω_c are all positive frequencies, then we obtain,

$$\gamma_a^{NL}(\underline{k}) = \frac{-(8\pi^2)^2}{\left.\frac{\partial Re\,\epsilon}{\partial\omega}\right|_{\omega=\omega_a}} \int d^3\underline{k}_1 \frac{d^3\underline{k}_2}{(2\pi)^3} \frac{|\chi|^2 I_c(\underline{k}_2)}{\left.\frac{\partial Re\,\epsilon}{\partial\omega_1}\right|_{\omega_1=+\omega_b(\underline{k}_1)}}$$

$$\left[\delta^3(\underline{k}-\underline{k}_1-\underline{k}_2)\delta(\omega_a+\omega_b-\omega_c)-\delta^3(\underline{k}+\underline{k}_1-\underline{k}_2)\delta(\omega_a-\omega_b-\omega_c)\right]. \tag{153}$$

We have used here the symmetry $\chi(K,-K_1,K_2)=\chi(K,K_1,K_2)$, which is evident from equation (109). (We are also assuming an isotropic plasma.) Note, there is no contribution from $\delta(\omega_a-\omega_1+\omega_c)$ because $\omega_a+\omega_c$ is a high frequency. To interpret (153) in terms of a "Golden-rule", we introduce the quantum occupation number $n_c(\underline{k}_2)$, related to the energy density $U(\omega_c)$ and the spectral function $I_c(k_2)$ by,

$$n_c(\underline{k}_2) \equiv \frac{U(\omega_c)}{\hbar\omega_c} = \frac{1}{\hbar}\left.\frac{\partial \mathrm{Re}\,\epsilon}{\partial\omega}\right|_{\omega=\omega_c(k_2)} I_c(k_2)/8\pi \tag{154}$$

Now we can write (153) as

$$\gamma_a^{NL}(\underline{k}) = \frac{\hbar}{2}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\,\delta^3(\underline{k}-\underline{k}_1-k_2)\,|M|^2\Big[\delta(\omega_a-\omega_c-\omega_b) - \delta(\omega_a-\omega_c+\omega_b)\Big]\,n_c(k_2) \tag{155}$$

The matrix element squared is

$$|M|^2 = 16\pi\,(2\pi)^3\,(8\pi^2)^2\,\frac{|\chi|^2}{\left.\frac{\partial\mathrm{Re}\,\epsilon}{\partial\omega}\right|_{\omega=\omega_a(k)}\left.\frac{\partial\mathrm{Re}\,\epsilon}{\partial\omega}\right|_{\omega=\omega_b(k)}\left.\frac{\partial\mathrm{Re}\,\epsilon}{\partial\omega}\right|_{\omega=\omega_c(k_2)}} \tag{156}$$

Equation (155) is part of the result which can be obtained by the "Golden-rule" methods of Tsytovich. The idea is to write down an equation for the time rate of change of bosons (e.g., plasmons), $n_b(\underline{k})$, using rules associated with certain diagrams for quantum processes and their inverses.

Consider the following processes:

Fig. 1

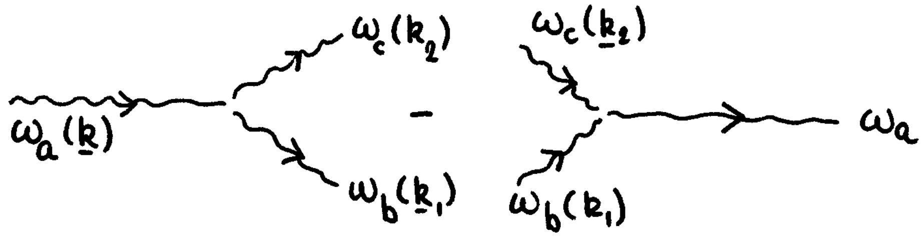
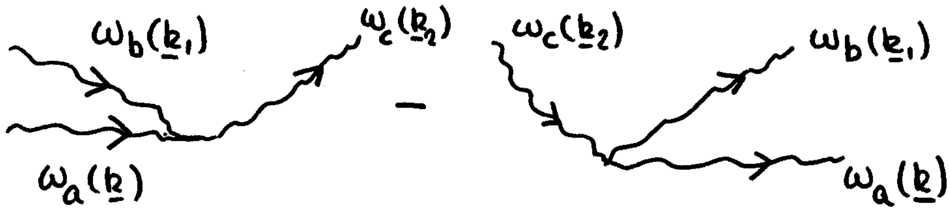


Fig. 2



In Figure 1, one has the decay or absorption of a boson "a", and the emission of the two bosons "b" and "c", which is a loss mechanism for n_a, and the inverse, which is a production mechanism. In Figure 2, one has the absorption of "b" and "a", and the emission of "c" which is also a loss mechanism for n_a, and the inverse, which is a gain mechanism. Since the probability of absorption of a boson is proportional to n and the probability of emission is proportional to $n+1$ (the 1 corresponds to spontaneous emission), the rate equation becomes

$$\frac{\partial n_a}{\partial t} = -\hbar \int \frac{d^3\underline{k}_1}{(2\pi)^3} \frac{d^3\underline{k}_2}{(2\pi)^3} |M|^2 \Big\{ \delta^3(\underline{k} - \underline{k}_1 - \underline{k}_2)\, \delta(\omega_a - \omega_b - \omega_c) \left[n_a (n_b+1)(n_c+1) - (n_a+1) n_b n_c \right] + \delta^3(\underline{k} + \underline{k}_1 - \underline{k}_2)\, \delta(\omega_a + \omega_b - \omega_c) \left[n_a n_b (n_c+1) - (n_a+1)(n_b+1) n_c \right] \Big\}. \tag{157}$$

The delta functions conserve energy and momentum. We have assumed the same matrix elements for both the processes in Figure 1 and in Figure 2. We can combine terms proportional to n_a, and express the rate equation in the form

$$\frac{\partial n_a}{\partial t} = -2\gamma_a n_a + S_a , \tag{158}$$

where γ_a is the loss-rate and S corresponds to spontaneous emission. Then,

$$-2\gamma_a = -\hbar \int \frac{d^3\underline{k}_1}{(2\pi)^3} \frac{d^3\underline{k}_2}{(2\pi)^3} |M|^2 \Big\{ \delta^3(\underline{k}-\underline{k}_1-\underline{k}_2)\,\delta(\omega_a-\omega_b-\omega_c) \left[n_b+n_c\right] + \delta^3(\underline{k}+\underline{k}_1-\underline{k}_2)\,\delta(\omega_a+\omega_b-\omega_c)\left[n_b-n_c\right] \Big\} \tag{159}$$

and

$$S = +\hbar \int \frac{d^3\underline{k}_1}{(2\pi)^3} \frac{d^3\underline{k}_2}{(2\pi)^3} |M|^2 \Big\{ \delta^3(\underline{k}-\underline{k}_1-\underline{k}_2)\,\delta(\omega_a-\omega_b-\omega_c) + \delta^3(\underline{k}+\underline{k}_1-\underline{k}_2)\,\delta(\omega_a+\omega_b-\omega_c) \Big\} n_b n_c . \tag{160}$$

In steady state, $S_a = -2\gamma_a n_a$, and this is the quantum statement of detailed balancing which corresponds to Kirchoff's law, (147) in the turbulent plasma. When the turbulence in the low frequency wave is low, n_b will be small, and equation (159) corresponds exactly to (155). Equation (160) exhibits the dependence on $I_b(k_1)\,I_c(k_2)$ which we have found in the nonlinear contribution to the spontaneous emission in equation (137).

LECTURE VII

June 25, 1974

A. REVIEW AND IDENTIFICATING OF STOKES AND ANTI-STOKES PROCESSES

We have found an expression for the damping rate of a wave with frequency, ω_a, due to turbulence at frequency ω_c acting as a "pump" to couple ω_a to a third low frequency, at $\pm\omega_b$. By making the resonant approximation (152) for $\operatorname{Im} \epsilon(K_1)^{-1}$, the expression (151) reduced to a difference of delta functions. From the relative signs in (153), we see that the term for which $\omega_a = \omega_c - \omega_b$ is destabilizing to wave "a", and the term for which $\omega_a = \omega_c + \omega_b$ is stabilizing. These correspond to the mode at ω_a playing the role of a Stokes or anti-Stokes line. The difference between the coherent and incoherent cases here is that the "pump" $I_c(\underline{k}_2)$ exists over a range of $\underline{k}_2$ values which include values satisfying both signs if the frequency-matching condition,

$$\omega_a(\underline{k}) - \omega_c(k_2^{\pm}) = \pm\,\omega_b\,(\underline{k} - \underline{k}_2^{\pm}) \tag{161}$$

We shall see in the case treated below, that $k_2^- > k_2^+$, so local stabilization requires that where the spectrum is large $I(k_2^-) < I(k_2^+)$ that is, the spectrum decreases as k_2 increases, locally. One approximation that is sometimes useful, essentially ignores the difference between k_2^- and k_2^+. Then the condition for local stabilization is that the derivative of I(k) with respect to k be negative. We call the approximation in which the turbulent damping rate is proportional to the negative-derivative of I with respect to k, the "derivative-approximation".

B. DERIVATIVE APPROXIMATION FOR THE TURBULENT DAMPING RATE

In equation (153) for the turbulent danping rate, $\gamma_a^{NL}(\underline{k})$, send $\underline{k}_1 \to -\underline{k}_1$ in the second integral, and assume an isotropic plasma, for which $\omega_b(-\underline{k}_1) = \omega_b(\underline{k}_1)$. That is, ω_b depends only on the magnitude k_1. Then, by expanding the difference of delta-functions in ω_b (assumed $\ll \omega_a$ or ω_b), we arrive at the following expression:

$$\gamma_a^{NL}(\underline{k}) = + \frac{2(8\pi^2)^2}{\left.\frac{\partial \operatorname{Re}\epsilon}{\partial\omega}\right|_{\omega_a}} \int \frac{d^3k_2}{(2\pi)^3} \left|x(\underline{k},\omega_a,\underline{k}_1,\omega_b,\underline{k}_2,\omega_c)\right|^2$$

$$\frac{I_c(\underline{k}_2)}{\left.\frac{\partial \operatorname{Re}\epsilon}{\partial\omega}\right|_{\omega_b(k_1)}} \omega_b(k_1) \frac{\partial}{\partial\omega_c(k_2)} \delta\left(\omega_c(k_2) - \omega_a(k)\right) \tag{162}$$

Here, $\underline{k}_1 \equiv \underline{k} - k_2$. Next, note the identity,

$$\frac{\partial}{\partial\omega_c}\delta(\omega_c - \omega_a) = \frac{1}{\frac{\partial\omega_c}{\partial k_2}} \frac{\partial}{\partial k_2}\left[\frac{1}{\left|\frac{\partial\omega_c}{\partial k_2}\right|}\delta(k_2 - \bar{k}_2)\right], \tag{163}$$

where $\bar{k}_2$ is the value of k_2 which satisfies,

$$\omega_c(\bar{k}_2) - \omega_a(k) = 0 \quad , \tag{164}$$

When waves "c" and "a" have the same linear dispersion relations, $\bar{k}_2 = k$. Let us specialize to this case, now, and, in fact, consider the case of Langmuir turbulence, with coupling to ion-acoustic waves.

Then,

$$\begin{aligned} \omega_a(k) = \omega_c(k) = \omega_L(k) &\equiv \omega_{pe}^2 + \frac{3}{2}\frac{V_e^2 k^2}{\omega_p}, \\ \omega_b(k_1) = \omega_{ia}(k_1) &\equiv c_s k_1 \end{aligned} \tag{165}$$

where C_s is the sound speed. It is also easy to prove from the linear dielectric functions that

$$\left.\frac{\partial \mathrm{Re}\,\epsilon}{\partial \omega}\right|_{\omega_L(k)} \approx \frac{2}{\omega_{pe}}\,,$$
$$\left.\frac{\partial \mathrm{Re}\,\epsilon}{\partial \omega}\right|_{\omega_{ia}(k)} \approx \frac{2}{\omega_{ia}(k)}\,\frac{k_D^2}{k_1^2} \tag{166}$$

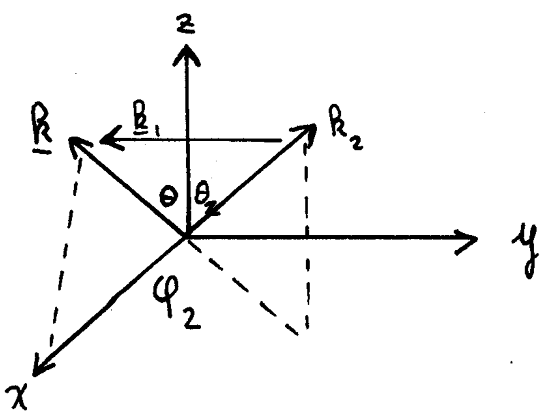
Finally, from our warm plasma evaluation,

$$x(\underline{k}_1, \omega_{ia}(k_1); \underline{k}_2, \omega_p; \underline{k}, \omega_p) \approx \frac{-ie^3 n_0 k_1 \hat{k}_2\cdot\hat{k}}{2m^2\omega_p^2 v_e^2 k_1^2} = \frac{-ie\,\hat{k}_2\cdot\hat{k}}{8\pi\theta_e k_1}\,, \tag{167}$$

Using the symmetry relations for x, equation (162) for the turbulent Langmuir damping rate becomes,

$$\gamma_L^{NL}(\underline{k}) = \frac{\pi\omega_p e^2}{\theta_e^2}\int\frac{d^3k_2}{(2\pi)^3}\,\frac{(\hat{\underline{k}}_2\cdot\hat{\underline{k}})}{k_1^2}\, I_L(\underline{k}_2)\left\{\frac{\pi}{2}\,\omega_{ia}^2(k_1)\,\frac{k_1^2}{k_D^2}\cdot\frac{1}{\frac{3v_e^2 k_2}{\omega_p}}\,\frac{\partial}{\partial k_2}\left[\frac{1}{\frac{3v_e^2}{\omega_p}k_2}\,\delta(k_2-k)\right]\right\}, \tag{168}$$

The factor in parenthesis, $\{\ \}$, represents our approximation here for $\epsilon^{-1}(\underline{k}-\underline{k}_2, \omega_L(\underline{k})-\omega_L(\underline{k}_2))$. To proceed further, we assume the turbulence is azimuthally symmetric about a direction which we take as the Z-axis of a spherical coordinate system in $\underline{k}_2$-space. We also choose the X-axis so that $\underline{k}$ lies in the X-Z plane:



$$\mu \equiv \cos\theta$$
$$\mu_2 \equiv \cos\theta_2$$

Then the turbulent Langmuir spectrum $I_L(\underline{k}_2) = I_L(k_2, \mu_2)$, is independent of the azimuthal angle φ_2, and (168) becomes

$$\gamma_L^{NL}(\underline{k}) = \frac{a}{2\pi} \int_0^\infty dk_2 \int_{-1}^{+1} d\mu_2 \int_0^{2\pi} d\varphi_2 (\hat{k}_2 \cdot \hat{k})^2 k_1^2 k_2 \, I_L(k_2, \mu_2) \frac{\partial}{\partial k_2} \frac{\delta(k_2 - k)}{k_2}, \tag{169}$$

where

$$a = \pi^3 \frac{\omega_p^3 e^2}{\theta_e^2 V_e^4} \frac{c_s^2}{9} \frac{1}{(2\pi)^3}, \tag{170}$$

If we now integrate by parts, this becomes

$$\gamma_L^{NL}(\underline{k}) = -a \int_{-1}^{+1} d\mu_2 \int_0^{2\pi} d\varphi_2 \frac{(\hat{k}_2 \cdot \hat{k})^2}{k} \frac{\partial}{\partial k_2} \left[k_1^2 k_2 \, I_L(k_2, \mu_2) \right] \Bigg|_{k_2 = k}, \tag{171}$$

In this coordinate system,

$$\underline{k} \cdot \underline{k}_2 = k k_2 \left[\sin\theta \sin\theta_2 \cos\varphi_2 + \cos\theta \cos\theta_2 \right], \tag{172}$$

and

$$k_1^2 = k^2 + k_2^2 - 2 k k_2 \left\{ \sin\theta \sin\theta_2 \cos\varphi_2 + \cos\theta \cos\theta_2 \right\}, \tag{173}$$

To proceed further, we assume the turbulence is symmetric about reflection in the x-y plane, so that I_L is an even function of μ_2. We next perform the φ_2 integral, which gives zero for all terms proportional to an odd power of $\cos\varphi_2$. If we also discard terms which vanish by the symmetry of I in μ_2, then the term $\{\ \}$, above, makes no contribution, and the result is,

$$\gamma_L^{NL}(k) = -\frac{a}{2} \int_{-1}^{+1} d\mu_2 \left[(1-\mu^2)(1-\mu_2^2) + 2\mu^2\mu_2^2\right] \frac{1}{k} \left[\frac{\partial}{\partial k_2} k_2 (k^2 + k_2^2) I_L(k_2)\right]_{k_2 = k},$$

(174)

The factor which depends on I_L, k, and k_2 can be written more concisely as

$$\frac{1}{k} \left[\frac{\partial}{\partial k_2} k_2 (k^2 + k_2^2) I_L(k_2, \mu_2)\right]_{k_2 = k} = 2 \frac{\partial}{\partial k} \left[k^2 I_L(k, \mu_2)\right], \tag{175}$$

Hence, our final expression is

$$\gamma_L^{NL}(k,\mu) = -a \frac{\partial}{\partial k} \int_{-1}^{+1} d\mu_2 \left[\mu_2^2(3\mu^2 - 1) + (1-\mu^2)\right] k^2 I(k, \mu_2), \tag{176}$$

C. SPHERICALLY-SYMMETRIC TURBULENCE

The above expression becomes particularly simple when the turbulence is spherically symmetric:

$$\gamma_L^{NL}(k) = -\frac{4}{3} a \frac{\partial}{\partial k} I(k), \tag{177}$$

In accord with our earlier discussion, the turbulent damping rate is proportional to the negative derivative of I with respect to k. Equation (177) is sometimes a useful approximation even when the spectrum is not spherically symmetric, or for one dimensional problems. For example, suppose we are looking for saturation of a cylindrically, symmetric linear Langmuir wave instability by seeking a solution of

$$\gamma_L(k,\mu) + \gamma_L^{NL}(k,\mu) = 0$$

We can average this equation over angles and define the angle-averaged linear growth rate by

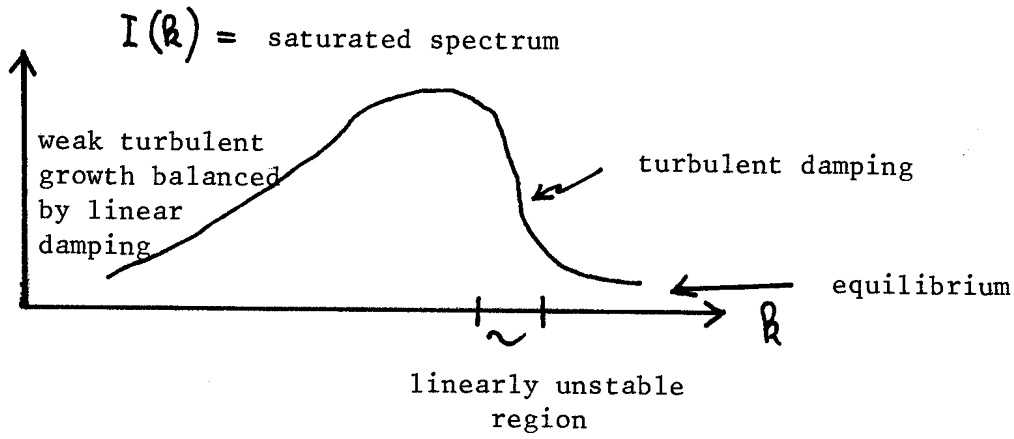
$$\overline{\gamma_L(k)} \equiv \frac{1}{2}\int_{-1}^{+1} d\mu \, \gamma_L(k,\mu)$$

The result is,

$$-\overline{\gamma_L(k)} = -\frac{2}{3}\, a \frac{\partial}{\partial k}\, \overline{I(k)} \quad , \tag{178}$$

where I(k) is the angle-averaged spectrum. (178) can often be integrated between k-values where the spectrum is linear.

Equation (178) or (176) is often said to represent a process of diffusion in k-space, with energy migrating to smaller k-numbers, where there is linear stability. In these regions, the slope can afford to change sign and the linear damping can balance a small amount of turbulent negative damping. For example, fluctuations which are linearly unstable in the interval indicated below, can spread to smaller k where the fluctuations are linearly damped.



D. STIMULATED SCATTERING FROM IONS (NONLINEAR LANDAU DAMPING OFF IONS)

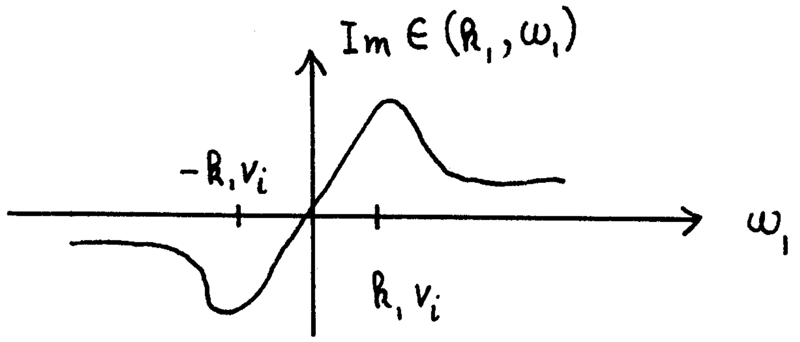
The process we have just described can be called stimulated scattering of Langmuir-waves from ion-acoustic waves. However, at equal temperatures, the ion-acoustic waves are not "good" normal modes because of heavy ion Landau damping. Nevertheless a similar turbulent damping exists, and can be treated by not making the resonant approximation for $-\mathrm{Im}\,\epsilon^{-1}(\underline{k}_1, \omega_a - \omega_c)$ about $\pm\,\omega_b(k_1)$. This so-called stimulated scattering from ions (or nonlinear Landau-damping off ions), can be thought of as the turbulence analogue of the quasi-mode instabilities discussed in Lecture IV, section B. The derivative approximation can again be justified, but this time by invoking the collisionless expression for the imaginary part of the linear susceptibility. The dominant contribution comes from the ions:

$$\mathrm{Im}\,\epsilon(k_1, \omega_a - \omega_c) =$$

$$-\frac{\pi\omega_{pi}^2}{n_0 k^2}\int d^3\underline{v}\;\underline{k}_1\cdot\frac{\partial f_{oi}}{\partial\underline{v}}\;\delta(\omega_a - \omega_c - \underline{k}_1\cdot\underline{v})$$

$$= \sqrt{\frac{\pi}{2}}\;\frac{k_{Di}^2}{k_1^2}\;\frac{(\omega_a - \omega_c)}{k_1 v_i}\;e^{-(\omega_a-\omega_c)^2/2v_i^2 k_1^2}\,, \tag{179}$$

Here $v_i^2 = \theta_i/m_i$, where θ_i is the ion temperature, and $k_{Di}^2 = 4\pi e^2 n_0/\theta_i$ is the inverse ion Debye length squared. This is an odd function of $\omega_a - \omega_c$, which has extrema near $\omega_a - \omega_c = \pm k_1 v_i$, and which can be approximated again as a derivative of a delta-function:



$$\operatorname{Im} \epsilon(k_1, \omega_a - \omega_c) \rightarrow -\pi \frac{k_{Di}^2}{k_1^2} k_1 v_i \frac{\partial}{\partial \omega_a} \delta(\omega_a - \omega_c), \tag{180}$$

From this, an expression similar to (162) follows, with a factor $|\epsilon(k_1, \omega_a - \omega_c)|^{-2}$ which is slowly varying (at equal temperatures) and can be absorbed into the matrix element or taken equal to its value at $\omega_a = \omega_c$. A turbulent damping rate expression formally similar to (176), but with a different constant "a" can again be obtained and has been applied in the saturation of linear instabilities such as the parametric instability. The derivative approximation always yields smooth turbulent spectra, because differentiation is a smoothing operation. If the dielectric function $\operatorname{Im} \epsilon^{-1}$ is not approximated by the derivative of a delta-function, a spiked structure results in the spectrum, in which the spikes are separated by multiples of ω_b or $k_1 v_i$. These problems generally require numerical solutions.

E. COHERENT WAVES IN A PERIODIC BACKGROUND

In the "parametric-approximation" to interactions of three coherent waves, we considered a known periodic (sinusoidal) "pump" field to be part of the field in terms of which perturbation theory was developed. The "background" field was assumed to be independent of space and time. An alternate method which is often useful is to consider a periodic field as part of the background plasma. In the Vlasov equation, this means it is kept in the Vlasov operator, $\mathcal{O}$, and the zero-order problem is to solve $\mathcal{O} f_0 = 0$, together with Maxwell's equations. The study of small wave perturbations in a periodic plasma is of interest to us for several reasons:

1. The parametric instability theory for three, four or even more interacting waves can often be treated more easily when the pump is part of the background, provided the zero-order problem can be solved. By working to first order in the self-consistent field but to all orders in the pump field, one can thereby find three and four-wave susceptibilities by expanding the exact susceptibilities which contain all orders of the pump field. It is also easier to calculate certain symmetries in parametric instabilities this way.

2. There are some parametric instabilities which essentially require all orders of the pump (are not expandible). An example is when the pump is a monochromatic large-amplitude Langmuir wave (sometimes called a Bernstein - Greene - Kruskal solution). This wave can trap electrons in its potential-energy troughs allowing the density of trapped electrons to thereby couple perturbing fields at different frequencies. This results in parametric instabilities which cannot be expanded in the pump, and can only be treated by considering the large-amplitude Langmuir wave to be part of the background plasma.

3. We can easily calculate the effects of a long-wavelength monochromatic pump on spontaneous Cerenkov emission, by considering the pump as part of the background. Some interesting effects occur here.

4. The theory of turbulence in a periodic background plasma can be developed when the periodicity arises from a monochromatic coherent field.

F. THE DIPOLE APPROXIMATION IN WARM FLUID THEORY

One of the simplest interesting examples is the calculation of the "linear" electrostatic susceptibility for small amplitude waves in a background warm plasma which includes an infinite* wavelength sinusoidal pump to all orders. For simplicity, we assume an isothermal equation of state. We must solve the fluid equations, which, for either species, are

$$\frac{\partial n}{\partial t} + \underline{V} \cdot \underline{\nabla} n + n \underline{\nabla} \cdot \underline{V} = 0$$

$$n \frac{\partial \underline{V}}{\partial t} + n \underline{V} \cdot \underline{\nabla} \underline{V} + \frac{1}{m} \gamma \Theta \underline{\nabla} n = \frac{q}{m} n E$$

We seak an expansion in which

$$\underline{E} = \underline{E}_0(t) + E_1$$

$$n = n_0 + n_1$$

$$\underline{V} = \underline{V}_0 + \underline{V}_1 ,$$

where,

$$\underline{E}_0(t) \equiv \underline{E}_0 \sin \omega_0 t$$

is a zero-order field which is assumed known. The zero-order equations then give

$$n_0 = \text{const}$$

$$\frac{\partial \underline{V}_0}{\partial t} = \frac{q}{m} \underline{E}_0(t) \text{ , or}$$

$$\underline{V}_0(t) = \underline{u}_0 \cos \omega_0 t \text{ , where } \underline{u}_0 = - \frac{q \underline{E}_0}{m \omega_0} \tag{181}$$

---

*This is called the dipole approximation by analogy with atomic theory when the radiation field has a long wavelength compared to atomic dimensions. In the plasma case, the wavelength of the pump must be longer than all the other wavelengths of interest.

The first-order equations, Fourier-transformed in space, and with the subscript one suppressed, are,

$$\frac{\partial n}{\partial t} + i\underline{v}_0 \cdot \underline{k}\, n + i n_0 \underline{k} \cdot \underline{v} = 0 \quad , \tag{182}$$

$$n_0 \frac{\partial}{\partial t} \underline{v} + i n_0 \underline{v}_0 \cdot \underline{k}\, \underline{v} + \frac{\gamma \theta}{m} i \underline{k}\, n = \frac{q}{m} n_0 \underline{E} \quad , \tag{183}$$

$\underline{v}$ can be eliminated from (182), by operating with

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + i\, \underline{k} \cdot \underline{v}_0 (t) \, . \tag{184}$$

on equation (182). The result is

$$\left[\frac{d^2}{dt^2} + k^2 V_s^2\right] n(\underline{k}, t) = i \frac{q}{m} n_0 \underline{k} \cdot \underline{E}(\underline{k}, t) \, , \tag{185}$$

where $V_s^2 \equiv \gamma_s \theta_s / m_s$.

This is easily transformed by introducing a new variable $\eta(t)$, defined by

$$n(t) = \eta(t)\, e^{-i \int_{t_0}^{t} \underline{k} \cdot \underline{v}_0 (t')\, dt'} \quad , \tag{186}$$

Then, (185) becomes

$$\left[\frac{\partial^2}{\partial t^2} + k^2 V_s^2\right] \eta = i \frac{q}{m} \underline{k} \cdot \underline{E}\, n_0 e^{i \int_{t_0}^{t} dt' \underline{k} \cdot \underline{v}_0 (t')} \quad , \tag{187}$$

But this is just the equation of a forced harmonic oscillator, and is solved by the following Green's function:

$$G_k(\tau) = \frac{1}{kv_s}\,\Theta(\tau)\sin kv_s\tau \ , \tag{188}$$

where, $\Theta(\tau)$ is the step-function, defined to be zero for $\tau < 0$ and one for $\tau > 0$. The solution for the density is

$$n(\underline{k},t) = i\,\frac{qn_0}{m}\int_{t_0}^{\infty} d\bar{t}\; G_k(t-\bar{t})\,\underline{k}\cdot\underline{E}(\underline{k},\bar{t})\,F(t,\bar{t}) \ , \tag{189}$$

where

$$F(t,\bar{t}) \equiv e^{\,i\int_t^{\bar{t}} dt'\,\underline{k}\cdot\underline{v}_0(t')} \ , \tag{190}$$

when $\underline{V}_0 \to 0$, $F \to 1$, and (189) becomes just the usual fluid result for the density response to an electrostatic field.

That is, we may rewrite (184) for the first-order charge density in a longitudinal field as,

$$p(\underline{k},t) \equiv qn(\underline{k},t) = -ik\int_{t_0}^{\infty} d\bar{t}\; x(\underline{k},t-\bar{t})\,E(\underline{k},\bar{t})\,F(t,\bar{t}) \ , \tag{191}$$

where,

$$x(\underline{k},t-\bar{t}) = \frac{q^2 n_0}{m}\,G_k(t-\bar{t}) \ , \tag{192}$$

The susceptibility is perhaps more familiar when Fourier transformed in time:

$$x(\underline{k},\omega) = -\,\frac{q^2 n_0}{m}\,\frac{1}{\omega^2 - k^2 v_s^2} \ , \tag{193}$$

In the presence of a periodic field, we must deal with the factor F. If we perform the integrals in the exponent, the result is,

$$F(t,\bar{t}) = e^{i\alpha \sin \omega_0 \bar{t}} \; e^{-i\alpha \sin \omega_0 t} , \tag{194}$$

where,

$$\alpha \equiv \frac{\underline{k} \cdot \underline{u}_0}{\omega_0} , \tag{195}$$

is proportional to the particle excursion distance in the pump, divided by the wavelength of the perturbing wave. It is now convenient to define a linear susceptibility in the periodic background plasma by

$$\chi_0(t-\bar{t}, \bar{t}) = \chi(t-\bar{t}) F(t-\bar{t}+\bar{t}, \bar{t}) \tag{196}$$

Considered as a function of the two variables $t-\bar{t}$ and $\bar{t}$, this is clearly a periodic function of $\bar{t}$, as we can see from the definition (194) of F .

We now have,

$$\rho(t) = -ik \int_{t_0}^{\infty} d\bar{t} \; \chi_0(t-\bar{t}, \bar{t}) E(\bar{t}) , \tag{197}$$

where k-dependence has been suppressed. This is a typical result for first order waves in a periodic background. When there is spatial periodicity, it parallels the time behaviour (i.e. χ_0 depends on $\underline{r} - \bar{\underline{r}}$, and is a periodic function of $\bar{\underline{r}}$, as well). A solution technique which is based on Floquet's theorem is generally valid in all such cases, and may be applied here. Since χ_0 is periodic in $\bar{t}$ we make the Fourier series expansion,

$$x_0(t-\bar{t},\bar{t}) = \sum_n x_n(t-\bar{t})\, e^{in\omega_0 \bar{t}} \quad , \tag{198}$$

If we then Fourier transform equation (197) in time, the result may be written as

$$\rho(\omega) = -ik\, x(\omega,\omega+n\omega_0)\, E(\omega+n\omega_0) \quad , \tag{199}$$

where

$$x(\omega,\omega+n\omega_0) \equiv x_n(\omega) \quad , \tag{200}$$

and we have applied summation convention in the repeated index, n. By then applying Poisson's equation,

$$ik\, E(\omega) = 4\pi\, (\rho_e + \rho_i) \quad , \tag{201}$$

one generates a set of matrix equations in frequency space given by

$$\epsilon(\omega+m\omega_0,\omega+n\omega_0)\, E(\omega+n\omega_0) = 0, \tag{202}$$

where

$$\epsilon(\omega+m\omega_0,\,\omega+n\omega_0) \equiv \delta_{nm} + 4\pi\, x(\omega+m\omega_0,\omega+n\omega_0) , \tag{203}$$

The infinite set of equations (202) is analogous to equation (67) for the 3-wave case. However, now the pump is allowed to interact with the field perturbations, E, an infinite number of times. It is important to note that these equations are <u>linear</u> in E. The pump again acts to couple say, $E(\omega)$ to $E(\omega-\omega_0)$ but we now see that this is not just proportional to $E_0 e^{i\omega_0 t}$, but can involve the pump in third order, $E_0^3 e^{i\omega_0 t} e^{i\omega_0 t} e^{-i\omega_0 t}$; in fifth order, $E_0^5 e^{i\omega_0 t} e^{i\omega_0 t} e^{-i\omega_0 t} e^{i\omega_0 t} e^{-i\omega_0 t}$, and so forth.

One can obtain an explicit expression for $\chi(\omega + m\omega_0, \omega + n\omega_0)$ in the above approximation, by using the Bessel function identity

$$e^{i\alpha \sin \omega_0 t} = \sum_\ell J_\ell(\alpha)\, e^{i\ell\omega_0 t} , \tag{204}$$

in both factors of (194) and then forming (196). The result is

$$\chi(\omega + m\omega_0 , \omega + n\omega_0) = \sum_{\ell=-\infty}^{+\infty} J_{\ell+m}(\alpha)\, J_{\ell+n}(\alpha)\, \chi(\omega - \ell\omega_0), \tag{205}$$

From this expression we see that the coupling coefficient in the periodic plasma is proportional to linear susceptibilities. The largest nonlinear response is proportional to the low frequency electron density response, when one of the frequencies, of the coupled waves is low. Suppose ω is the low frequency, then the dominant terms in (205) occur at $\ell = 0$, and when the pump parameter $\alpha = \underline{k} \cdot \underline{u}_0 / \omega_0$ is $\ll 1$, the expression reduces to

$$\chi(\omega + m\omega_0 , \omega + n\omega_0)\Big|_{\substack{\omega \ll k v_e \\ n,m > 0}} \approx J_m(\alpha)\, J_n(\alpha)\, \frac{k_{De}^2}{k^2}$$

$$\approx \frac{1}{m!\,n!} \left(\frac{\alpha}{2}\right)^{m+n} \left[1 - \frac{1}{4}\frac{\alpha^2}{(n+1)!}\right]\left[1 - \frac{1}{4}\frac{\alpha^2}{(m+1)!}\right], \tag{206}$$

When n and/or m is negative, we must use the Bessel function identity $J_{-n} = (-1)^n J_n$ to find the correct overall sign in (206). We can immediately verify that $4\pi E_0 \chi^m(\omega, \omega - \omega_0) = -(\alpha/2)(k_{De}^2/k^2)$ agrees with the result from equations (69) and (109), if we remember that the phase factor φ in (59) and (61) must be set equat to $-\pi/2$ to agree with the present choice of phase.

LECTURE VIII

June 27 , 1974

A. GENERAL FORMULATION SMALL-AMPLITUDE WAVE PROPERTIES IN A PERIODIC BACKGROUND PLASMA

The formal results obtained last time from the warm fluid dipole approximation can be shown to be generally valid (e.g. from the Vlasov equation). That is, the Fourier transform of the "linear" coherent current in a background plasma with periodicity $K_0 = (\omega_0, \underline{k}_0)$ arising from, say, a monochromatic pump field, is

$$j(K) = -i\omega \sum_n \chi(K, K+nK_0) E(K+nK_0) , \tag{207}$$

where

$$j(K) = \hat{\underline{e}}(K) \cdot \underline{j}(K) , \tag{208}$$

and

$$\chi(K, K+nK_0) \equiv 4\pi \, \hat{e}_\ell(K) \, \chi_{\ell m}(K, K+nK_0) \hat{e}_m(K+nK_0),$$

and χ contains all orders of the background pump field. Note the current is <u>linear</u> in the self-consistent field E. The expression we derived last time, in equation (205), from fluid theory for the electrostatic susceptibility in an infinite wavelength pump can also be shown to be precisely correct for an infinite wavelength pump in the Vlasov equation, provided the χ's in (205) are understood as the collisionless linear susceptibility. No new information is gained, however, since the small electron Landau-damping in $\mathrm{Im}\, \chi_e(\omega)$ at low frequencies is completely negligible, compared to the reactive contribution, $Re\ \chi_e(\omega) = k_D^2/k^2$. When the background pump is electrostatic, there is a contribution to $\chi(K, K+nK_0)$ which comes from electrons trapped in the potential energy troughs of the pump which is not expandible in powers of the coherent pump field. For the present we shall not treat this case.

If we make the linear diagonal approximation and insert (207) into Maxwell's equation for E, the result is

$$\epsilon(K+nK_0, K+mK_0)\, E(K+mK_0) = 0, \tag{209}$$

where,

$$\epsilon(K+nK_0, K+mK_0) \equiv \delta_{nm} + \chi(K+nK_0, K+mK_0), \tag{210}$$

We have here assumed no external currents and electrostatic fields. In general, the susceptibility $\chi(K+nK_0, K+mK_0)$ consists of an electron and an ion contribution, but it is often only necessary to retain the ion contribution in the leading (linear) diagonal terms. When electromagnetic waves are involved*, we must rewrite equation (210) as

$$\epsilon(K+nK_0, K+mK_0) \equiv \frac{c^2 k_n k_m}{\omega^2} + \left(1 - \frac{c^2 k^2}{\omega^2}\right)\delta_{nm}$$

$$+ \chi(K+nK_0, K+mK_0), \tag{211}$$

---

*When a mixture of electrostatic and electromagnetic waves at the same frequency is involved, it is best to put additional labels on the scalar fields and ϵ's to identify then.

Equation (209) is an infinite set of equations, coupled in Fourier-space. They may be solved when the background field is not too large by the same truncation methods we used to reduce equations (64)-(66) to a soluble (2x2) form (62) in the "three-wave" case. The essence of the truncation approximation is that the new-normal modes $E(K)$ be near two or three linear frequencies, even though the susceptibilites $\chi(K, K-K_0)$, or $\chi(K, K-2K_0)$ may contain the pump to all orders. One then solves a 2x2 or 3x3 homogeneous matrix problem for the parametrically renormalized new-normal modes, which may be unstable. The coupled equations in (209) still represent the "parametric-approximation", because the background field is a parametric which is assumed known.

B. SYMMETRIES IN THE PARAMETRIC APPROXIMATION

It is easy to prove some symmetries that we previously assumed, by now using the fluid, dipole, approximate result of equation (205). These results can also be proven generally from the Vlasov equation, without making the dipole (infinite pump wavelength) approximation. First note from (205), that χ is a symmetric matrix

$$\chi(\omega+m\omega_0,\ \omega+n\omega_0) = \chi(\omega+n\omega_0, \omega+m\omega_0) \tag{212}$$

This agrees with the symmetry in (67) when we recall that $E^0_m = \mathbb{E}^0_m e^{-i\pi/2}$ since then $\chi(K, K-K_0)$ in (67) is purely real.

Next, we can prove that if ω is a new normal mode, so that

$$\det\ \epsilon(\omega+m\omega_0, \omega+n\omega_0) = 0,$$

then $\omega_0-\omega^*$ is also a new normal mode, in an isotropic plasma.

Since ϵ is a linear superposition of linear susceptibilities, each of which satisfies $\epsilon(-\omega^*)^* = \epsilon(\omega)$, it follows that

$$\det \epsilon(-\omega^* + [m+1]\omega_0, -\omega^* + [n+1]\omega_0) =$$
$$\det \epsilon(\omega - [m+1]\omega_0, \omega - [n+1]\omega_0)^*$$

Since the matrices are infinite, we can send $-(m+1) \rightarrow m$, and $-(n+1) \rightarrow n$, which proves the desired result.

C. FOUR-WAVE INTERACTIONS

A typical parametric instability due to four-wave interactions is the so-called oscillating two-stream instability. In addition to the pump, we retain a linear wave at ω, one at $\omega - \omega_0$, and one at $\omega + \omega_0$. Suppose ω a low frequency, and $\mathrm{Re}\,\omega \mp \omega_0$ are close to linear Langmuir frequencies, $\mp \omega_L$. We consider the pump to have infinite wavelength, so that the other three waves have the same wave number, k. Making the 3x3 truncation procedure on (209), yields the matrix equation,

$$\begin{pmatrix} \epsilon(\omega-\omega_0, \omega-\omega_0), & \chi(\omega-\omega_0, \omega), & \chi(\omega-\omega_0, \omega+\omega_0) \\ \chi(\omega, \omega-\omega_0), & \epsilon(\omega, \omega), & \chi(\omega, \omega+\omega_0) \\ \chi(\omega+\omega_0, \omega-\omega_0), & \chi(\omega+\omega_0, \omega), & \epsilon(\omega+\omega_0, \omega+\omega_0) \end{pmatrix} \begin{pmatrix} E(\omega-\omega_0) \\ E(\omega) \\ E(\omega+\omega_0) \end{pmatrix} = 0, \tag{213}$$

Next, we assume a convergent expansion in the pump field exists $(|\alpha| << 1)$ in equation (206), so that $\epsilon(\omega \mp \omega_0, \omega) \sim E_0$ $\epsilon(\omega - \omega_0, \omega + \omega_0) \sim E_0^2$, and $\epsilon(\omega, \omega)$ consists of a linear part $1 + \chi_e(\omega) + \chi_1(\omega)$, and a second order part $\chi^{NL}(\omega) \sim E_0^2$. If we now take the determinant of the matrix in (213), and use the symmetry (212), the new normal modes are the solution to

$$
\begin{aligned}
0 = \det\ & \epsilon(\omega + n\omega_0, \omega + m\omega_0) \\
\approx\ & \epsilon(\omega - \omega_0, \omega - \omega_0)\, \epsilon(\omega, \omega)\, \epsilon(\omega + \omega_0, \omega + \omega_0) \\
-\ & \epsilon(\omega - \omega_0, \omega - \omega_0) \left[\chi(\omega, \omega + \omega_0)\right]^2 \\
-\ & \epsilon(\omega + \omega_0, \omega + \omega_0) \left[\chi(\omega, \omega - \omega_0)\right]^2 ,
\end{aligned}
\tag{214}
$$

Here we have discorded terms proportional to E_0^3 and E_0^4, by eliminating $\chi(\omega \mp \omega_0, \omega \pm \omega_0)$ (off-diagonal) contributions to the determinant. The diagonal terms $\epsilon(\omega \pm \omega_0; \omega \pm \omega_0)$ and $\epsilon(\omega, \omega)$ still make contributions which are higher-order in the pump, since

$$
\begin{aligned}
\epsilon(\omega \pm \omega_0, \omega \pm \omega_0) &= \epsilon(\omega \pm \omega_0) + \chi^{NL}(\omega \pm \omega_0) \\
\epsilon(\omega, \omega) &= \epsilon(\omega) + \chi^{NL}(\omega) ,
\end{aligned}
\tag{215}
$$

where $\epsilon(\omega \pm \omega_0)$ and $\epsilon(\omega)$ are linear, and χ^{NL} is second-order in the pump. These are sometimes called self-energy corrections, since, they do not involve shifts in frequency or coupling between Fourier coefficients $E(\omega)$. If we use (215) and work only to second-order in the pump, the result is,

$$0 = \epsilon(\omega-\omega_o)\,\epsilon(\omega)\,\epsilon(\omega+\omega_o) + \epsilon(\omega-\omega_o)\,\epsilon(\omega+\omega_o)\,\chi^{NL}(\omega)$$
$$+ \epsilon(\omega-\omega_o)\,\epsilon(\omega)\,\chi^{NL}(\omega+\omega_o) + \epsilon(\omega+\omega_o)\,\epsilon(\omega)\,\chi^{NL}(\omega-\omega_o)$$
$$- \epsilon(\omega-\omega_o)\left(\chi(\omega,\omega+\omega_o)\right)^2 - \epsilon(\omega+\omega_o)\left(\chi(\omega,\omega-\omega_o)\right)^2, \tag{216}$$

The solutions of équation (216) include the parametric 4-wave instabilities (3 normal modes coupled by pump). To proceed further with them, requires explicit expressions for the coupling coefficients and self-energy coefficients, from equation (206). Before doing this, we note that the 3-wave instabilities are included in (214) in the special case when $\epsilon(\omega+\omega_0)$ is sufficiently far off resonance (large) that only its coefficient in (214) need vanish. The vanishing of the coefficient of $\epsilon(\omega+\omega_0)$ in (214) then gives precisely the condition (70) for 3-wave instabilities, when self-energies are also ignored. This condition is that $|\mathrm{Re}\,\omega+\omega_0-\omega_L(k)| \gg \gamma_L(k)$, or roughly, that $2\,\mathrm{Re}\,\omega \gg \gamma_L(k)$. When the low frequency in 3-wave interactions gets too small, a fourth wave at $\omega+\omega_0$ must always be taken into account. An example is the oscillating two-streams instability, in which $\omega \approx 0$.

D. OSCILLATING TWO-STREAM INSTABILITY

From (206), we find the leading terms when $|\alpha| \ll 1$ are

$$\chi^{NL}(\omega) = -\alpha^2/4$$
$$\chi^{NL}(\omega\pm\omega_o) = \alpha^2/4 \tag{217}$$
$$\chi(\omega,\omega\pm\omega_o) = \pm\frac{\alpha}{2}\frac{k_D^2}{k^2}$$

If we insert into (216), the result may be written as

$$0=\epsilon(\omega-\omega_0)\,\epsilon(\omega+\omega_0)+\frac{\alpha^2}{4}\Big[\epsilon(\omega-\omega_0)+\epsilon(\omega+\omega_0)\Big]\left[1-\frac{k_{De}^2}{k^2\epsilon(\omega,\omega)}\right], \quad (218)$$

We have lumped the self-energy term $\chi^{NL}(\omega)$ back together with $\epsilon(\omega)$ in $\epsilon(\omega,\omega)$, for reasons soon to be apparent.

We now make resonant approximation for $\epsilon(\omega\pm\omega_0)$, as follows,

$$\begin{aligned}\epsilon(\omega\pm\omega_0)&=\pm\frac{2}{\omega_p}\left(\omega\pm\omega_0\mp\omega_L(k)+i\,\gamma_L(k)\right)\\&=\pm\frac{2}{\omega_p}\left(\omega\pm\delta+i\,\gamma_L\right)\quad ,\end{aligned} \quad (219)$$

where we have defined

$$\delta=\omega_0-\omega_L \quad (220)$$

δ is a kind of frequency mismatch factor, as we shall see later. Inserting (219) into (218), the dispersion relation becomes,

$$-\frac{4}{\omega_p^2}\left[(\omega+i\gamma)^2-\delta^2\right]+\alpha^2\frac{\delta}{\omega_p}\left[1-\frac{k_{De}^2}{k^2\epsilon(\omega,\omega)}\right]=0\,, \quad (221)$$

This is a quadratic equation in ω , provided $|\omega|\ll k v_i$, so that $\epsilon(\omega,\omega)=1+k_{De}^2/k^2+k_{Di}^2/k^2+\mathcal{O}(\alpha^2)$. The solution is

$$\omega=-i\,\gamma\pm\delta\sqrt{1+\frac{\alpha^2}{4}\,\frac{\omega_p}{\delta}\,\mu}\ , \quad (222)$$

where

$$\mu\equiv\frac{\Theta_e}{\Theta_i+\Theta_e} \quad (223)$$

is the ratio of the electron to the electron plus ion temperatures. It is clear from equation (223), that for $\delta < 0$ and sufficiently large,

$$\frac{\alpha^2 \omega_p \mu}{4|\delta|} > 1 \quad , \tag{224}$$

the frequency ω is purely imaginary, and one sign corresponds to pure growth. This is called the purely-growing or oscillating two-stream instability, depending on whether the emphasis is upon the mode ω or the modes $\omega \pm \omega_0$. The threshold condition is

$$\alpha^2 = - \frac{4(\gamma^2 + \delta^2)}{\omega_p \mu \delta} \quad , \tag{225}$$

and the minimum threshold occurs when

$$\delta = -\gamma \quad ,$$
$$\alpha^2 \Big|_{min} = \frac{8}{\mu} \frac{\gamma}{\omega_p} \tag{226}$$

This is on the same order as the electron-ion decay instability for equal electron and ion temperatures. The growth rate close to the minimum threshold is given by

$$\gamma_g \Big|_{max} = \gamma \left(\frac{\alpha^2}{\alpha^2/_{min}} - 1 \right) \quad , \tag{227}$$

We note that this is second order in the pump, as is the growth rate, (83), for the electron-ion decay instability. Thus, the 4-wave parametric instability is only higher-order than the associated 3-wave instability in the sense of a third-order current being necessary to obtain the self-energy terms in (215). This suggests that a complete turbulence theory to order I^2 also requires self-energy corrections from j^{f3}, and this is indeed the case, as we shall see later. Right now we also note that the form of the dispersion relation in (218), as derived from (216) relied on the following combination of terms of order α^2:

$$\cdot\ x^{NL}(\omega \pm \omega_o) - \frac{[x(\omega, \omega \pm \omega_o)]^2}{\epsilon(\omega, \omega)} \quad , \tag{228}$$

This combination often occurs, and is necessary also to describe certain quasi-mode parametric instabilities, such as stimulated Compton scattering, self-focusing, and nonlinear Landau damping from electrons.

E. PARAMETRIC INSTABILITIES WITH COHERENT TRAPPING EFFECTS

Consider a one-dimensional steady state in which there is a background monochromatic traveling electrostatic wave whose potential is $\varphi_o(x', t) = \varphi_o \cos(k_o x' - \omega_o t)$. In the wave frame, $x = x' + \omega_o t / k_o$, and the electrons and ions obey a time-independent Vlasov equation. Since the background is then time-independent, the zero-order problem is

$$v \frac{\partial f_o}{\partial x} - \frac{q}{m} \frac{\partial \varphi_o(x)}{\partial x} \frac{\partial f_o}{\partial v} = 0 \quad , \tag{229}$$

$$-\frac{\partial^2 \varphi_o(x)}{\partial x^2} = 4\pi \int_{-\infty}^{+\infty} dv \left[-e f_{oe}(x, v) + e f_{oi}(x, v) \right] , \tag{230}$$

The stationary potential is $\varphi_o(x) = \varphi_o \cos k_o x$. Equation (229) may be written in terms of Lagrangian coordinates ("characteristics") as

$$\frac{d}{dt'} f_o(x_o(t'), v_o(t')) = 0 \quad , \tag{231}$$

where,

$$\ddot{x}_o(t') = -\frac{q}{m} \left. \frac{\partial \varphi}{\partial x} \right|_{x = x_c(t')} \quad , \tag{232}$$

is the equation of motion of a particle in the background field, and

$$V_0(t') = \frac{\partial x_0(t')}{\partial t'} \quad , \tag{233}$$

Equation (231) has any solutions in which f_0 is constant along an orbit $x_0(t')$, $V_0(t')$. A first integral of (232) gives the usual result of energy conservation,

$$\frac{m\,v_0(t)^2}{2} + q\,\varphi_0 \cos k_0 x_0(t) = W \quad , \tag{234}$$

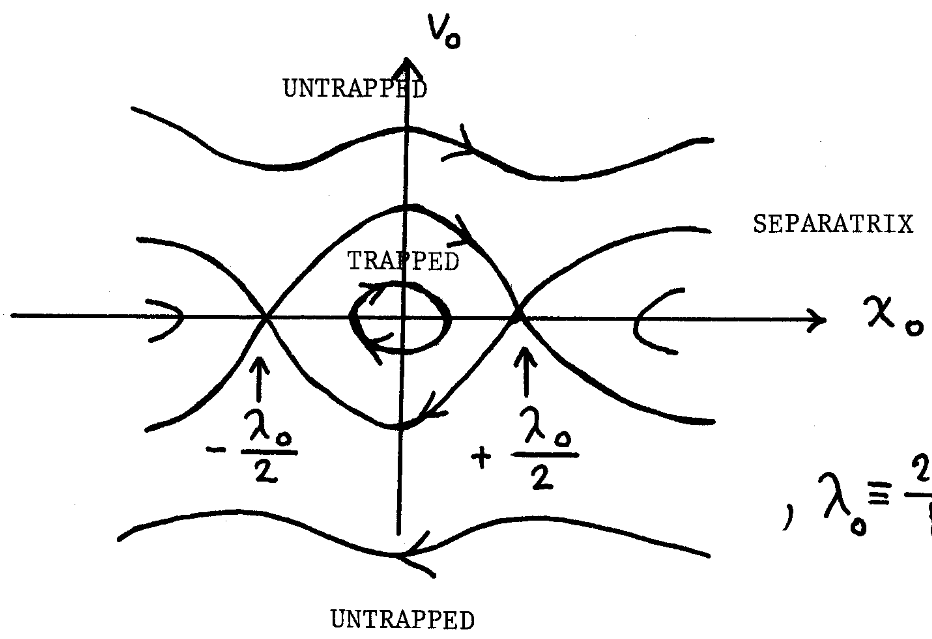
where W is the energy of a trajectory. There are two kinds of orbits, which can be classified by energy. These are those whose energy satisfies,

$$q\varphi\Big|_{min} < W < q\varphi\Big|_{max} \quad , \tag{235}$$

in which the particles are trapped in the potential energy troughs, and those for which

$$q\varphi\Big|_{max} < W \quad , \tag{236}$$

in which particles do not reverse the net sign of their velocity. The orbits are most easily visualized in phase space ($x_0 - V_0$ space)



$$, \ \lambda_0 \equiv \frac{2\pi}{k_0}$$

The untrapped orbits are identified by the value of the energy, W $(>q\varphi|_{max})$, and the sign of the velocity. The trapped orbits are identified by the energy W $(<q\varphi|_{max})$ alone. If we visualize the distribution function as plotted "above" phase space, on a third axis, then the statement that the distribution function must be constant along an orbit is that it can depend only on the energy, and, for untrapped electrons, on the sign of the velocity. Thus, for either species,

$$f_0 = \Theta(v)\, f^{>}(W) + \Theta(-v)\, f^{<}(W)\,, \tag{237}$$

$$f^{>} = f^{<} \quad \text{in trapped region only} \tag{238}$$

The stationary solution in (237) and (238) results because the moving particles are assumed uniformly distributed in number along their orbits. If we now form the charge density and switch from velocity to energy, as an integration variable, then

$$\rho(x) = \frac{q}{\sqrt{2m}} \int_{q\varphi}^{\infty} \frac{dW}{\sqrt{W-q\varphi}} \left[f^{<}(W) + f^{>}(W)\right], \tag{239}$$

With the assumed potential $\varphi_0(x)$, we may make an arbitrary choice of $f^{<}$ and $f^{>}$ for the trapped and untrapped ions, and for the untrapped electrons, and can then regard Poisson's equation as an integral equation for the trapped electron distribution function. In general this method leads to an infinite number of distribution function f_{0e} which are consistent with the assumed periodic potential. These solutions are often called Bernstein-Greene-Kruskal equilibria, after the authors who first pointed out their existence. O'Neil has discussed which of these is an accessible nonlinear state for a large amplitude electrostatic wave in a Maxwellian plasma.

The stability of such a periodic solution can be tested by studying small electrostatic perturbations, and our general remarks about susceptibilities hold for this system. From the Vlasov equation

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{q}{m}\frac{\partial \varphi_0}{\partial x}\frac{\partial f_1}{\partial v} = -\frac{q}{m}E_1(x,t)\frac{\partial f_0}{\partial v}, \tag{240}$$

we may solve for the charge density to first order in E, and all orders in φ_0 (see Goldman, Phys.Fluids 13, 1281 (1970)). The result, in the laboratory frame is

$$\chi(k+\ell k_0, k+m k_0) = \chi^T_{\ell m} + \chi^u_{\ell m} \tag{241}$$

where the contribution (for either species) from the trapped and untrapped particles are given by,

$$\chi^T_{\ell m} = -\frac{4\pi e^2}{m}\sum_{n=1}^{\infty}\int_{q\varphi|_{min}}^{q\varphi|_{max}} dW \frac{F_{\ell mn}(W)}{(\omega-kv_0)^2 - n^2\omega_T(\omega)^2}$$

$$\chi^u_{\ell m} = -\frac{4\pi e^2}{m}\sum_{n=-\infty}^{+\infty}\int_{q\varphi|_{max}}^{\infty} dW \left[\frac{F^{>}_{\ell mn}(W)}{\omega-kv_0-\{k+[\ell+n]k_0\}u(W)} - \frac{F^{<}_{\ell mn}(W)}{\omega-kv_0+\{k+[\ell+n]\}u(W)}\right], \tag{242}$$

Here, $V_0 \equiv \omega_0/k_0$, so the combination $\omega - kv_0$ is the Doppler-shifted frequency, and the "oscillator-strengths", $F_{\ell mn}$ for trapped and untrapped particles are functions only of the energy W and the particulary periodic "equilibrium" distribution function, f_0. $\omega_T(W)$ and $U(W)$ are, respectively, the bounce frequency for trapped particles, and the average velocity of untrapped particles:

$$\omega_T(W)^{-1} \equiv \frac{2}{\pi}\left(\frac{m}{2}\right)^{1/2} \int_0^{x_W} dx \, \frac{1}{\sqrt{W - q\varphi_0(x)}} , \tag{243}$$

$$U(W)^{-1} \equiv \frac{1}{\lambda_0}\left(\frac{m}{2}\right)^{1/2} \int_{-\lambda_0/2}^{+\lambda_0/2} dx \, \frac{1}{\sqrt{W - q\varphi_0(x)}} , \tag{244}$$

Here x_W is the turming point for a trapped particle, and $\lambda_0 \equiv 2\pi/k_0$. The susceptibilities in (242) are in the form of a classical sum over oscillator contributions. Interesting resonance effects can occur when $(\omega - kv_0) = \pm n\,\omega_T(W)$. The trapped particle susceptibility cannot be expanded in term of the field strength φ_0 because of such resonances and because of (243). Since the matrix element $\chi^T_{\ell m}$ has sensitive frequency dependence, it can contribute to parametric instabilities in ways we have not seen thus for. For example, a coupled stokes wave at ω and anti-stokes wave at $\omega - 2\omega_0$ can be driven unstable by an electrostatic pump. ω turns out to be down-shifted from ω_0 by a bounce frequency, ω_T , and $2\omega_0 - \omega$ up-shifted by the same amount. Such parametric instabilities are often called "sideband instabilities", and have been observed experimentally by Wharton, Malmberg and O'Neil.

F. FOUR-WAVE INTERACTIONS IN TURBULENCE

For completeness, we will now indicate the form of four-wave interactions for turbulence in a stationary homogeneous background. If we add a scalar contribution to the right side of equation (134) which corresponds to the fluctuating third-order current in (51c), this takes the form

$$-4\pi i\, j^{3f} = -4\pi\omega \int dK_{123}\, x(K\, K_1 K_2 K_3)$$

$$\left[\mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3 - \mathcal{E}_1 \langle \mathcal{E}_2 \mathcal{E}_3 \rangle - \langle \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3 \rangle \right] \tag{245}$$

The procedure decrived in part A of Lecture VI can then be carried out with this additional term, and the random phase approximation applied directly to

$$\langle \mathcal{E}(K')\, \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3 \rangle - \langle \mathcal{E}(K')\, \mathcal{E}_1 \rangle \langle \mathcal{E}_2 \mathcal{E}_3 \rangle$$

There is no additional contribution to the spontaneous emission, but the renomalized dielectric function $\tilde{m}$, now takes the following form, rather than the form in (142):

$$\tilde{m}(K) = m(K) - 4(4\pi)^2 \omega \int dK_{12}\, \frac{|x(K\, K_1 K_2)|^2\, I(K_2)}{\tilde{m}(K_1)/\omega_1}$$

$$+ 8\pi\omega \int dK_2\, x(K_1 - K_2, K_2, K)\, I(K_2) \quad , \tag{246}$$

The additional term in (246) is essentially a turbulent self-energy correction. The two nonlinear terms in (246) give rise to turbulent analogues of oscillating two-stream instabilities, which are called modulational interactions and can be important for very long wavelength turbulence. They also give rise to turbulent analogues of stimulated scattering from electrons (non-linear Compton scattering), and self-focusing instabilities.

B I B L I O G R A P H Y

Much of the unpublished work presented in these lectures has developed out of a long collaboration with D.F. DuBois, whom we gratefully acknowledge. The following references may prove of interest as supplementary reading:

KLIMONTOVICH FORMALISM

Y.L. Klimontovich, "The Statistical Theory of Non-Equilibrium Processes in a Plasma" (M.I.T. Press, 1967)

TURBULENCE AND QUASILINEAR THEORY

B.B. Kadomtsev "Plasma Turbulence" (Academic Press, 1965)

V.N. Tsytovich "Plasma Turbulence" (Academic Press, 1965)

V.N. Tsytovich "Nonlinear Effects in Plasma" (Plenum Press, 1970)

Sagdeev and Galeev "Nonlinear Plasma Theory" (Benjamin, 1969)

COHERENT PARAMETRIC INSTABILITIES

V.P. Silin, JETP 21, 1127 (1965)

D.F. DuBois and M.V. Goldman, Phys.Rev.Lett. 14, 544 (1965)

K. Nishikawa, J.Phys.Soc.Jap. 24, 916, 1152 (1968)

LINEAR INCOHERENT THEORY OF PARAMETRIC INSTABILITIES

M.V. Goldman, Am.Phys. (N.Y.) 38, 95 (1966)

D.F. DuBois and M.V. Goldman, Phys.Rev. 164, 207 (1967)

D.F. DuBois, Statistical Physics of Charged Particle Systems, R.Kubo and T. Kihara, ed. (Syokabo, Tokyo, Benjamin, 1969)

NONLINEAR THEORY OF PARAMETRIC INSTABILITIES

D.F. DuBois and M.V. Goldman, Phys.Rev.Lett. 28, 218 (1972)

E. Valeo, C. Oberman and F.W. Perkins, Phys.Rev.Lett. 28, 340 (1972)

D.F. DuBois and M.V. Goldman, Phys.Fluids 15, 919 (1972)

M.V. Goldman and D.F. DuBois, "Nonlinear Laser Heating of a Plasma", Kirtland Air Force Base Technical Report, AFWL-TR-21-101 Sept. 1972

COHERENT TRAPPING PHENOMENA

T.M. O'Neil, Phys.Fluids 8, 2255 (1965)

C.B. Wharton, J.H. Malmberg and T.M. O'Neil, Phys.Fluids 11, 1761 (1968)

W. Kruer, J. Dawson and R. Sudan, Phys.Rev.Lett. 23, 838 (1969)

M.V. Goldman, Phys.Fluids 13, 1281 (1970)

Sagdeev and Galeev, "Nonlinear Plasma Theory", pgs 37-43 (Benjamin, 1969).

E R R A T A

The following is a list of some errors in the foregoing eight lectures:

1. The "proof" in Lecture B, part C, of the symmetry relations (55), (56) and (57) is not correct, although these symmetries are properties of the model 3-wave susceptibility given in equation (107). The proof fails, as pointed out by Professor Weibel, because the parts of $\chi_{\ell mn}(K\,K_1\,K_2)$ which do <u>not</u> have the symmetry (56) integrate to zero in the integral (53).

2. The statements on pages (50) and (51) that the expression (104) is correct in a D.C. magnetic field with the proper definition of $\underline{\underline{U}}$ appear to be false, since use was mode of the form of $\underline{\underline{U}}$ in (105) when deriving (104). However, equation (97) is still an appropriate starting point for deriving the susceptibility in a magnetic field, with $\underline{\underline{T}}$ redefined in term of the linear conductivity, in the presence of a magnetic field.

3. The right side of equation (69) should be multiplied by 4π .

4. In equation (79), the first parentheses should have $+\Delta$ instead of $-\Delta$.

5. In equation (125), replace m(K) by $m(-K) = m(K)^*$.

6. The sign of the second term on the right side of (129) should be + rather than − .